Revista Integración, temas de matemáticas
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 37, N° 1, 2019, pág. 77–111

DOI: http://dx.doi.org/10.18273/revint.v37n1-2019005

On the existence of a priori bounds for positive solutions of elliptic problems, I

Rosa Pardo*
Universidad Complutense de Madrid, Departamento de Análisis Matemático y Matemática Aplicada, Madrid, Spain.

Abstract. This paper gives a survey over the existence of uniform L^∞ *a priori* bounds for positive solutions of subcritical elliptic equations

$$(\mathcal{P})_p \qquad -\Delta_p u = f(u), \text{ in } \Omega, \qquad u = 0, \text{ on } \partial\Omega,$$

widening the known ranges of subcritical nonlinearities for which positive solutions are *a-priori* bounded. Our arguments rely on the moving planes method, a Pohozaev identity, $W^{1,q}$ regularity for $q > N$, and Morrey's Theorem. In this part I, when $p = 2$, we show that there exists *a-priori* bounds for classical, positive solutions of $(\mathcal{P})_2$ with $f(u) = u^{2^*-1}/[\ln(e+u)]^\alpha$, with $2^* = 2N/(N-2)$, and $\alpha > 2/(N-2)$. Appealing to the Kelvin transform, we cover non-convex domains.

In a forthcoming paper containing part II, we extend our results for Hamiltonian elliptic systems (see [22]), and for the p-Laplacian (see [10]). We also study the asymptotic behavior of radially symmetric solutions $u_\alpha = u_\alpha(r)$ of $(\mathcal{P})_2$ as $\alpha \to 0$ (see [24]).
Keywords: *A priori* estimates, subcritical nonlinearity, moving planes method, Pohozaev identity, critical Sobolev hyperbola, biparameter bifurcation.

MSC2010: 35B45, 35J92, 35B33, 35J47, 35J60, 35J61.

Sobre la existencia de cotas a priori para soluciones positivas de problemas elípticos, I

Resumen. Este artículo proporciona un estudio sobre la existencia de cotas *a priori* uniformes para soluciones positivas de problemas elípticos subcríticos

$$(\mathcal{P})_p \qquad -\Delta_p u = f(u), \text{ en } \Omega, \qquad u = 0, \text{ sobre } \partial\Omega,$$

* E-mail: rpardo@mat.ucm.es
Received: 15 November 2018, Accepted: 20 December 2018.
To cite this article: R. Pardo, On the existence of a priori bounds for positive solutions of elliptic problems, I, *Rev. Integr. temas mat.* 37 (2019), No. 1, 77–111. doi: 10.18273/revint.v37n1-2019005.

ampliando el rango conocido de no-linealudades subcríticas para las que las soluciones positivas están acotadas *a priori*. Nuestros argumentos se apoyan en el método de 'moving planes', la identidad de Pohozaev, resultados de regularidad en $W^{1,q}$ para $q > N$, y el Teorema de Morrey. En esta parte I, cuando $p = 2$ demostramos que existen cotas *a priori* para soluciones positivas clásicas de $(\mathcal{P})_2$ con $f(u) = u^{2^*-1}/[\ln(e+u)]^\alpha$, siendo $2^* = 2N/(N-2)$, y para $\alpha > 2/(N-2)$. Consideramos también dominios no-convexos, recurriendo a la transformada de Kelvin.

En un siguiente artículo, parte II, extendemos nuestros resultados para sistemas elípticos Hamiltonianos (ver [22]) y al p-Laplacian (ver [10]). También estudiamos el comportamiento asintótico de las soluciones radialmente simétricas $u_\alpha = u_\alpha(r)$ de $(\mathcal{P})_2$ cuando $\alpha \to 0$ (ver [24]).

Palabras clave: Estimaciones *a priori*, no-linealidades subcríticas, método de 'moving planes', igualdad de Pohozaev, hipérbola crítica de Sobolev, bifurcación biparamétrica.

1. Introduction

We focus our attention on the following question: Under what growth conditions on f, the nonnegative solutions to the Dirichlet problem will be uniformly bounded? *A priori* bounds in the L^∞-norm of positive solutions provided a great deal of information, and it is a longstanding open problem.

In this paper, we provide sufficient conditions for having a-priori L^∞ bounds for a classical positive solutions to the boundary value problem

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded C^2 domain, and f is a subcritical nonlinearity.

For $N = 2$, Turner proved the following result. Let Ω be a simply connected domain in $\mathbb{R}^2$ with C^2 boundary, let f be a continuous real-valued function on $\Omega \times \mathbb{R}$, and let us consider

$$-\Delta u(x) = f(x, u), \quad x \in \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

If there are numbers p, $A, B > 0$ and $C \geq 0$ such that $1 < p < 3$, and $Au^p \leq f(x,u) \leq \max(BC^p, Bu^p)$ for $u \geq 0$, then all such solutions are *a priori* bounded for some constant $C = C(\Omega, p, A, B, C)$. For $1 < p < 2$, an analogous result holds when Δ is replaced by a more general elliptic operator. In case of radial symmetry, an analogous result holds for any $p > 1$, if $Au^p \leq f(u)$ for $u \geq C$, and $f(u) \leq \max(BD^p, Bu^p)$ for some $D \geq 0$ and all $u \geq 0$ (see [27] and also [11, Theorem 1.1]). Brezis and Turner in [4] allow a more general nonlinearity: $f = f(x, u, \nabla u)$ with smaller growth, $f(\cdot, u, \cdot)/u^{\frac{N+1}{N-1}} \to 0$ as $u \to \infty$.

When $N > 2$, the exponent $2^\star - 1 = \frac{N+2}{N-2}$ of a nonlinearity $f(s) = s^{2^\star-1}$ is critical from the viewpoint of Sobolev embedding; observe that $2^\star = \frac{2N}{N-2}$, and the embedding $H^1(\Omega)$

in $L^{2^\star}(\Omega)$ is not compact. Pohozaev proved that problem (1) does not have a solution if Ω is starshapped (see [25]), and Bahri-Coron, and Ding proved that problem (1) has a solution if Ω has non trivial topology in a certain sens, including some classes of non star-shaped domains and in particular the case of rings (see [2], [12]).

If

$$\lim_{s\to\infty} \frac{f(s)}{s^{2^\star-1}} = +\infty,$$

then problem (1) is supercritical. Consider

$$\left\{ \begin{array}{rcll} -\Delta u & = & \lambda f(u), & \text{in } B, \\ u & = & 0, & \text{on } \partial B, \end{array} \right. \tag{2}$$

where B is the unit ball, and

$$f(u) = (1+u)^q, \qquad \text{for} \quad q > \frac{N+2}{N-2}, \qquad \lambda \in \mathbb{R}.$$

Joseph and Lundgren for balls in $\mathbb{R}^N$, $N \geq 3$, provided sufficient conditions guaranteeing that (2) has an unbounded sequence of positive solutions (see [19]). Their results are obtained by a careful analysis involving phase plane and qualitative arguments.

If

$$\lim_{s\to\infty} \frac{f(s)}{s^{2^\star-1}} = 0,$$

the problem is of subcritical nature. The discussion given so far suggests that the subcritical growth of f is a necessary condition for the existence of *a priori* bounds for solutions to (1).

Nussbaum obtain *a priori* bounds for positive radial solutions in the *subcritical radial* case, when there exist some $\delta > 0,\ s_0 \geq 0$ such that $2NF(s) - (N-2)sf(s) \geq \delta sf(s)$ for $s \geq s_0$. Here $F(t) := \int_0^t f(s)\,ds$, see [23]. Observe that this hypothesis covers the case when $f(s) = \lambda(1+|s|)^q$ for some $q < \frac{N+2}{N-2}$. Consider $f(s) = s^{2^\star-1-\varepsilon}$ for $\varepsilon > 0$. It is well known that problem (1) has a solution u_ε (see P. L. Lions [21] and references therein). Atkinson and Peletier for balls in $\mathbb{R}^3$, and Han for the minimum energy solutions in non-spherical domains, proved that there exists $x_0 \in \Omega$ and a sequence u_ε such that $\lim_{\varepsilon\to 0} u_\varepsilon = 0$ in $C^1(\Omega \setminus \{x_0\})$ and $\lim_{\varepsilon\to 0} |\nabla u_\varepsilon|^2 = C\delta_{x_0}$ in the sense of distributions, where δ is the Dirac distribution, and C depends on N and on the best Sobolev constant in $\mathbb{R}^N$ (see [1], [18]).

A-priori bounds for subcritical nonlinearities on general domains were raised by Gidas and Spruck in [16] as well as by Figueiredo, Lions and Nussbaum in [11]. The *blow-up* method together with Liouville type theorems for solutions in $\mathbb{R}^N$ and in the half space $\mathbb{R}^N_+$, was introduced by Gidas and Spruck for nonlinearities essentially of the type $f(x,s) = h(x)s^p$, with $p \in (1, 2^\star - 1)$ and $h(x)$ continuous and strictly positive. De Figueiredo, Lions and Nussbaum [11] obtained a similar result using a different method. In convex domains in particular, it is based on the monotonicity results by Gidas, Ni and Nirenberg [14], obtained by using the Alexandrov-Serrin *moving plane* method [26], (which provides *a priori* bounds in a neighborhood of the boundary), on the Pohozaev identity ([25]) and on the L^p theory for Laplace equations given by Calderón-Zygmund and Agmon, Douglis and Niremberg estimates (see [17]). They extend some of the results to non-convex smooth domains through the Kelvin transform.

Their results assume on f the following condition:

$$\liminf_{s\to+\infty} \frac{\theta F(s) - sf(s)}{s^2 f(s)^{2/N}} \geq 0, \qquad \text{for some} \quad \theta \in [0, 2^\star).$$

They conjecture that this condition is not necessary, but it is essential in their proof. It can be see n that for $f_1(s) = s^{2^\star-1}/\ln(s+2)^\alpha$ with $\alpha > 0$,

$$\liminf_{s\to+\infty} \frac{\theta F_1(s) - sf_1(s)}{s^2 f_1(s)^{2/N}} = -\infty, \qquad \text{for any} \quad \theta \in [0, 2^\star),$$

where $F_1(s) = \int_0^s f_1(t)\,dt$, (see [5, Remark 2.3]). We prove the existence of apriori bounds when $f(s) = s^{2^\star-1}/\ln(s+2)^\alpha$, with $\alpha > 2/(N-2)$ (see Theorem 1.1).

Next we include several subsections to describe our *a priori* bounds results on semilinear elliptic equations, and on some non-convex regions. We leave the proofs for the following sections.

1.1. Semilinear elliptic equations

We state the existence of a-priori bounds for classical positive solutions of elliptic equations (1) when $f(u) = \frac{u^{2^\star-1}}{[\ln(e+u)]^\alpha}$, with $\alpha > \frac{2}{N-2}$, and $\Omega \subset \mathbb{R}^N$ is a bounded, convex C^2 domain (see Corollary 2.2 in [5]).

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary.*

Let us consider the BVP

$$\begin{cases} -\Delta u = \dfrac{u^{\frac{N+2}{N-2}}}{\ln(e+u)^\alpha}, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with $\alpha > 2/(N-2)$.

Then, there exists a uniform constant C, depending only on Ω and f, such that for every classical solution $u > 0$, to (3),

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

This Theorem is in fact a Corollary of Theorem 2.1 (see Subsection 2.3 for a proof of Theorem 2.1; see also [5, Corollary 2.2]). The ideas of the proof of Theorem 2.1 lie on the following arguments:

Step 1. The *moving planes method* provides L^∞ bounds in a neighborhood of the boundary for classical positive solutions of (1).

Step 2. *Pohozaev identity* relates some integral defined on Ω with some integral defined on the boundary. This equality, combined with bounds in a neighborhood of the boundary, give us a uniformly bounded integral in Ω.

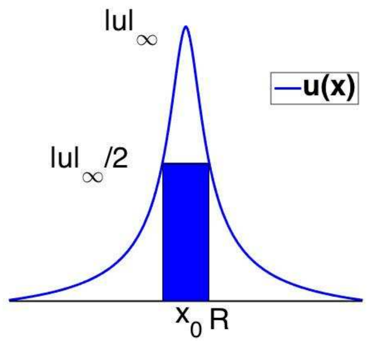


Figure 1. A solution u of (1), its L^∞ norm, and the estimate of the radius R such that $u(x) \geq \frac{\|u\|_\infty}{2}$ for all $x \in B(x_0, R)$, where x_0 is such that $u(x_0) = \|u\|_\infty$.

Step 3. The bounded integral in Ω previously obtained through Pohozaev identity, help us in lowering some $L^q(\Omega)$ bound of $f(u(\cdot))$. *Elliptic* $W^{2,q}$*-regularity* with $q \in (\frac{N}{2}, N)$ and *Sobolev embeddings* provide us $W^{1,q}$ bounds, with $q > N$. Through *Morrey's Theorem*, we estimate the radius R of a ball where the function u exceeds half of its L^∞ bound, see fig. 1.

Step 4. We reason by contradiction assuming that there exists an unbounded sequence of solutions $\{u_k\}$. *Elliptic* $W^{2,q}$*-regularity* with $q \in (\frac{N}{2}, N)$ and *Sobolev embeddings* provide us $W^{1,q}$ bounds with $q > N$, depending on k.

Step 5. Through *Morrey's Theorem*, we estimate the radius R_k of a ball where the function u_k exceeds half of its L^∞ bound, depending on k.

Step 6. Using this estimate we get a lower bound of the above uniformly bounded integral obtained in Step 2, reaching a contradiction, and deriving L^∞ bounds for classical positive solutions of (1).

The moving planes method was used earlier by Serrin in [26]. Gidas, Ni and Nirenberg characterized regions inside of Ω, where a positive solution cannot have critical points (see [14], [15]). They pose the following problem (see [14, p. 223]): *Suppose $u > 0$ is a classical solution of (1). Is there some $\delta > 0$ only dependent on the geometry of Ω (independent of f and u) such that u has no stationary points in a δ-neighborhood of $\partial\Omega$?* This is true in convex domains, and for $N = 2$. If f satisfies (H1) de Figueiredo, Lions and Nussbaum show us that there are some C and $\delta > 0$ depending only on the geometry of Ω (independent of f and u) such that

$$\max_{\Omega} u \leq C \max_{\Omega_\delta} u \tag{4}$$

where $\Omega_\delta := \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}$, (see [11] and Theorem A.11). Moreover, if f also satisfies (H4), then there exists a constant C depending only on Ω and f but not on u, such that

$$\max_{\Omega \setminus \Omega_\delta} u \leq C \tag{5}$$

(see [11] and Theorem A.12).

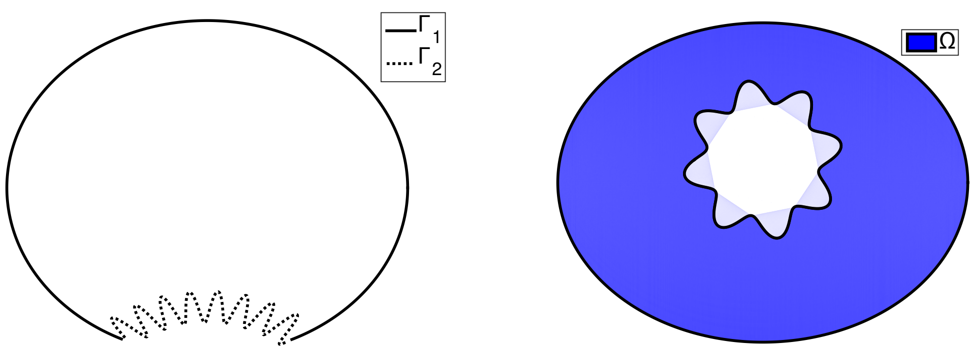
1.2. Ring-like regions

In Theorem 2.1 and Theorem 2.2 it is assumed either the monotonicity of $f(s)/s^{2^*-1}$ or the convexity of Ω respectively.

What about problems in non-convex domains or with nonlinearities that do not satisfy the monotonicity of $f(s)/s^{2^*-1}$? Building on the *a priori* estimates previously established, we obtain *a priori* estimates for classical solutions to elliptic problems with Dirichlet boundary conditions on regions with *convex-starlike* boundary. This includes *ring-like* regions.

We will say that a domain Ω has a convex-starlike boundary if $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \subset \partial\Omega_1$, for some convex domain $\Omega_1 \subset \mathbb{R}^N$, and $n(x) \cdot (x - y) < 0$ for some $y \in \mathbb{R}^N$ and for all $x \in \Gamma_2$. Here $n(x)$ denotes the outward normal to the boundary $\partial\Omega$, see fig. 2 (a).

A particular case appears when $\Omega = \Omega_1 \setminus \Omega_2$ with $\overline{\Omega}_2 \subset \Omega_1$, where Ω_1 is convex, and Ω_2 *star-like*, that is $n_2(x) \cdot (x - y) > 0$, for some $y \in \mathbb{R}^N$, and for all $x \in \partial\Omega_2$. Here $n_2(x)$ denotes the outward normal to the boundary $\partial\Omega_2$. In that case, we will say that Ω is a *ring-like* domain, see fig. 2 (b). Since (1) is invariant under translations, without loss of generality, we may assume $y = 0$; in other words, we may assume Ω_2 to be star-like with respect to zero.



(a) (b)

Figure 2. (a) A *convex-starlike* boundary. (b) A *ring-like* domain.

Theorem 1.2. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain with convex-starlike boundary. Let us consider the BVP*

$$\begin{cases} -\Delta u = \dfrac{u^{2^*-1}}{\ln(e+u)^{\alpha}}, & in \ \ \Omega, \\ u = 0, & on \ \ \partial\Omega, \end{cases} \tag{6}$$

with $\alpha > 2/(N-2)$.

Then, there exists a uniform constant C, depending only on Ω and f, such that for every classical solution $u > 0$ to (6),

$$\|u\|_{L^\infty(\Omega)} \le C.$$

Proof. It is a Corollary of Theorem 3.1 (see also [8, Theorem 2]). For this particular type of nonlinearities, this result is included in Theorem 1.1. But in the abstrac setting, Theorem 3.1 is not included in Theorem 2.1, because we do not assume $f(s)/s^{2^*-1}$ to be nonincreasing. ☑

For the proof of Theorem 3.1, we first prove *a priori* bounds near the convex part of the boundary going back to [11]. Using that the boundary term in the Pohozaev identity on the boundary of a star-like region does not change sign, the proof is concluded.

This paper is organized in the following way. In Section 2 we state and prove our abstract main theorem on *a priori* bounds for semilinear elliptic equations. In Section 3 we state and prove one abstract theorem on *a priori* bounds in a class of non convex domains.

We also collect some results on the *a priori* bounds in a neighborhood of the boundary in two Appendices. In Appendix A we describe the moving planes method, and its consequences when applied to a solution in a convex domain (see Theorem A.8). In Appendix B we apply the moving plane methods on the Kelvin transform, and its consequences for the general case (see Theorem A.12). All those results are essentially well known (see [11]). We include them for the sake of completeness and in order to make precise statements clarifying which hypothesis are needed in the convex case and in the non-convex case.

2. *A priori* bounds for semilinear elliptic equations

We provide *a-priori* $L^\infty(\Omega)$ bounds for a classical positive solutions to the boundary value problem (1), where $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded C^2 domain, and f is a subcritical nonlinearity.

Our main result in this Section are the following two theorems. The first one is on general smooth domains. The proof can be read in [5], we include it by the sake of completeness.

Theorem 2.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary. Assume that the nonlinearity f is locally Lipschitzian and satisfies the following conditions:*

(H1) $\dfrac{f(s)}{s^{2^\star-1}}$ *is nonincreasing for any $s > 0$.*

(H2) *There exists a constant $C_1 > 0$ such that* $\displaystyle\limsup_{s\to\infty} \frac{\max_{[0,s]} f}{f(s)} \le C_1.$

(H3) *There exists a constant $C_2 > 0$ and a non-increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(H3.1)
$$\liminf_{s\to+\infty} \frac{2NF(s)-(N-2)sf(s)}{sf(s)H(s)} \geq C_2 > 0,$$

and

(H3.2)
$$\lim_{s\to+\infty} \frac{f(s)}{s^{2^\star-1}\,\big[H(s)\big]^{\frac{2}{N-2}}} = 0.$$

(H4) $\liminf_{s\to+\infty} \frac{f(s)}{s} > \lambda_1$, *where* λ_1 *is the first eigenvalue of* $-\Delta$ *acting on* $H_0^1(\Omega)$.

Then, there exists a uniform constant C, *depending only on* Ω *and* f, *such that for every classical solution* $u>0$ *to* (1),
$$\|u\|_{L^\infty(\Omega)} \leq C.$$

If the domain Ω is convex, we have the following result:

Theorem 2.2. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded, convex domain with* C^2 *boundary. Assume that the nonlinearity* f *is locally Lipschitzian, satisfies* (H2)-(H4), *and also the following conditions:*

(H1)' *There exists a constant* $C_0>0$ *such that* $\liminf_{s\to\infty} \frac{\min_{[s/2,s]} f}{f(s)} \geq C_0$.

Then, there exists a uniform constant C, *depending only on* Ω *and* f, *such that for every classical solution* $u>0$ *to* (1),
$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Our analysis extends previous results, widen the known ranges of subcritical nonlinearities for which positive solutions are apriori bounded and also applies to non-convex domains.

All those results are known (see [5]). We include the proofs for the sake of completeness. Our proofs of Theorem 2.1 and Theorem 2.2, as in [11], use *moving plane arguments*, the *Kelvin transform*, and a Pohozaev identity (see [25]). These ideas are well known but we combine them in a slightly different way.

The moving planes method was used by Serrin in [26]. Gidas, Ni and Nirenberg in [14], using this moving planes method and the Hopf Lemma, prove symmetry of positive solutions of elliptic equations vanishing on the boundary. See also Castro-Shivaji [9], where symmetry of nonnegative solutions is established for $f(0)<0$. In [14] the authors also characterized regions inside Ω, next to the convex part of the boundary, where a positive solution cannot have critical points. Those regions, called *maximal caps*, depend only on the local convexity of Ω, and are independent of f and u (see the Appendix A.2 for a precise definition of maximal cap). This non-existence of critical points in a maximal cap, is due to the strict monotonicity of any positive solution in the normal direction. This is a key point to reach local *a priori* bounds in a neighborhood of the boundary.

The arguments split into two ways, depending on the convexity of the domain. The reason is the following one. If Ω is convex, and the nonlinearity f satisfies (H4), then any positive solution is *a priori* bounded in a neighborhood of the boundary; more precisely, there exists a constant C depending only on Ω and f but not on u, such that (5) holds (see [11] and Theorem A.8).

If Ω is a general bounded domain, not necessarilly convex, the argument on the *a priori* bounds in a neighborhood of the boundary relies on the Kelvin transform. In that case, if the nonlinearity f satisfies (H1) and (H4), then any positive solution is *a priori* bounded in a neighborhood of the boundary, in other words, conclusion (5) is reached, (see [11] and Theorem A.12). We include this Theorems in Appendix A and B in order to clarify which hypothesis are needed in the convex case and in the non-convex case respectively. The starting point in the proof of Theorems 2.1 and 2.2 are *a priori* bounds in a neighborhood of the boundary (Theorems A.12 and A.8, respectively).

In [6] and [7] we study the associated bifurcation problem for a nonlinearity $\lambda u+g(u)$ with g subcritical. We provide sufficient conditions guarantying that either for any $\lambda<\lambda_1$ there exists at least a positive solution, or for any continuum (λ, u_λ) of positive solution, there exists a $\lambda^*<0$ such that $\lambda^*<\lambda<\lambda_1$ and

$$\|\nabla u_\lambda\|_{L^2(\Omega)}\to\infty, \qquad \text{as}\quad \lambda\to\lambda^*$$

(see [7, Theorem 2]). In case Ω is convex, for any $\lambda<\lambda_1$ there exists at least a positive solution (see [6, Theorem 1.2]).

2.3. Proof of Theorems 2.1 and 2.2

Let us start this Subsection with the following remark.

Remark 2.3. By hypothesis, $H:\mathbb{R}^+\to\mathbb{R}^+$ is a non-increasing function, therefore $0\le \lim_{s\to\infty} H(s)<\infty$.

By hypothesis (H3.2) we also conclude that $\lim_{s\to+\infty}\frac{f(s)}{s^{2^*-1}}=0$.

Next, we prove Theorem 2.2 (we recall the ideas collected on Subsection 1.1).

Proof of Theorem 2.2. Step 1. From (5) and de Giorgi-Nash type Theorems (see [20, Theorem 14.1]),

$$\|u\|_{C^{0,\alpha}(\Omega_{\delta/8}\setminus\Omega_{7\delta/8})}\le C, \qquad \text{for any}\quad \alpha\in(0,1),$$

where $\Omega_t:=\{x\in\Omega\ :\ d(x,\partial\Omega)>t\}$.

From Schauder interior estimates (see [17, Theorem 6.2]),

$$\|u\|_{C^{1,\alpha}(\Omega_{\delta/4}\setminus\Omega_{3\delta/4})}\le C.$$

Finally, combining L^p estimates with Schauder boundary estimates (see [3], [17]),

$$\|u\|_{W^{2,p}(\Omega\setminus\Omega_{\delta/2})} \leq C, \qquad \text{for any} \quad p \in (1,\infty).$$

Consequently, there exists two constants $C, \delta > 0$ independent of u such that

$$\|u\|_{C^{1,\alpha}(\Omega\setminus\Omega_{\delta})} \leq C, \qquad \text{for any} \quad \alpha \in (0,1).$$

Step 2. From hypothesis (H3.1), there exists a constant $C_3 > 0$ and a non-increasing function H such that

$$2NF(s) - (N-2)sf(s) \geq \frac{C_2}{2}\, sf(s)H(s), \qquad \text{for any} \quad s > C_3.$$

Applying this inequality to any positive solution, and integrating on Ω, we obtain that

$$2N\int_\Omega F(u)\,dx - (N-2)\int_\Omega uf(u)\,dx \geq \frac{C_2}{2}\int_\Omega uf(u)H(u)\,dx - C_4, \tag{7}$$

for some constant C_4 independent of u. From now on, throughout this proof C denotes several constants independent of u.

From a slight modification of Pohozaev identity (see [11, Lemma 1.1] and [25]), if $y \in \mathbb{R}^N$ is a fixed vector, then any positive solution u of (1) satisfies

$$\int_{\partial\Omega} (x-y)\cdot n(x)\,|\nabla u|^2\,dS = 2N\int_\Omega F(u)\,dx - (N-2)\int_\Omega uf(u)\,dx. \tag{8}$$

This, (8) and (7) yield

$$\int_\Omega uf(u)H(u)\,dx \leq C, \tag{9}$$

for some constant C independent of u. Next we prove that also

$$\int_\Omega u|f(u)|\,H(u)\,dx \leq C. \tag{10}$$

From hypothesis (H4), there exists a constant C such that if $s > C$ then $f(s) > 0$. Therefore, splitting the above integral in the set $S = \{x \in \Omega : |u| \leq C\}$ and its complementary $\Omega \setminus S$, since from (9) $\int_{\Omega\setminus S} uf(u)H(u)\,dx \leq C$, then (10) holds.

Step 3. From hypothesis (H3.2), $\lim\limits_{s\to+\infty} \dfrac{|f(s)|^{\frac{1}{2^\star-1}}}{s\big[H(s)\big]^{\frac{2}{N+2}}} = 0$. Multiplying numerator and denominator by $|f(s)|H(s)^{\frac{N}{N+2}}$, we can assert that there exists a constant C such that

$$|f(s)|^{1+\frac{1}{2^\star-1}}\big[H(s)\big]^{\frac{N}{N+2}} \leq s|f(s)|H(s) + C, \qquad \text{for any} \quad s > 0.$$

Applying this inequality to any positive solution, integrating on Ω, and using (10) we obtain that

$$\int_\Omega |f(u)|^{1+\frac{1}{2^\star-1}} H(u)^{\frac{N}{N+2}}\,dx \leq C.$$

Consequently, since H is non-increasing,

$$\begin{aligned}
&\int_\Omega \big|f\big(u(x)\big)\big|^q\,dx \\
\leq\ & \frac{1}{H\big(\|u\|_\infty\big)^{\frac{N}{N+2}}}\int_\Omega \big|f\big(u(x)\big)\big|^{1+\frac{1}{2^\star-1}}\,H(u)^{\frac{N}{N+2}}\ \big|f\big(u(x)\big)\big|^{q-1-\frac{1}{2^\star-1}}\,dx \\
\leq\ & C\,\frac{\big\|f\big(u(\cdot)\big)\big\|_\infty^{q-1-\frac{1}{2^\star-1}}}{H\big(\|u\|_\infty\big)^{\frac{N}{N+2}}},
\end{aligned}\tag{11}$$

for any $q > N/2$.

Therefore, from elliptic regularity (see [17, Lemma 9.17]),

$$\|u\|_{W^{2,q}(\Omega)} \leq C\|\Delta u\|_{L^q(\Omega)} \leq C\,\frac{\big\|f\big(u(\cdot)\big)\big\|_\infty^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}}{\Big[H\big(\|u\|_\infty\big)\Big]^{\frac{N}{(N+2)q}}}.\tag{12}$$

Let us restrict $q \in (N/2, N)$. From Sobolev embeddings, for $1/q^* = 1/q - 1/N$ with $q^* > N$ we can write

$$\|u\|_{W^{1,q^*}(\Omega)} \leq C\|u\|_{W^{2,q}(\Omega)} \leq C\,\frac{\big\|f\big(u(\cdot)\big)\big\|_\infty^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}}{\Big[H\big(\|u\|_\infty\big)\Big]^{\frac{N}{(N+2)q}}}.$$

From Morrey's Theorem (see [3, Theorem 9.12 and Corollary 9.14]), there exists a constant C only dependent on Ω, q and N such that

$$|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^{1-N/q^*}\|u\|_{W^{1,q^*}(\Omega)}, \quad \forall x_1, x_2 \in \Omega.$$

Therefore, for all $x \in B(x_1, R) \subset \Omega$,

$$|u(x) - u(x_1)| \leq C\ R^{2-\frac{N}{q}}\|u\|_{W^{2,q}(\Omega)}.\tag{13}$$

Step 4. From now on, we shall argue by contradiction. Let $\{u_k\}_k$ be a sequence of classical positive solutions to (1) and assume that

$$\lim_{k\to\infty}\|u_k\| = +\infty, \qquad \text{where} \quad \|u_k\| := \|u_k\|_\infty.$$

Let $C, \delta > 0$ be as in (5). Let $x_k \in \overline{\Omega_\delta}$ be such that

$$u_k(x_k) = \max_{\Omega_\delta} u_k = \max_\Omega u_k.$$

By taking a subsequence if needed, we may assume that there exists $x_0 \in \overline{\Omega_\delta}$ such that

$$\lim_{k\to\infty} x_k = x_0 \in \overline{\Omega_\delta}, \qquad \text{and} \quad d_0 := dist(x_0, \partial\Omega) \geq \delta > 0.$$

Let us choose R_k such that $B_k = B(x_k, R_k) \subset \Omega$, and

$$u_k(x) \geq \frac{1}{2}\ \|u_k\| \qquad \text{for any} \quad x \in B(x_k, R_k).$$

and there exists $y_k \in \partial B(x_k, R_k)$ such that

$$u_k(y_k) = \frac{1}{2}\ \|u_k\|. \tag{14}$$

Let us denote by

$$m_k := \min_{[\|u_k\|/2, \|u_k\|]} f, \qquad M_k := \max_{[0,\|u_k\|]} f.$$

Therefore, we obtain

$$m_k \leq f\big(u_k(x)\big) \quad \text{if } x \in B_k, \qquad\qquad f\big(u_k(x)\big) \leq M_k \quad \forall x \in \Omega. \tag{15}$$

Then, reasoning as in (11), we obtain

$$\int_\Omega \big|f\big(u_k\big)\big|^q\ dx \quad \leq \quad C\ \frac{{M_k}^{q-1-\frac{1}{2^\star-1}}}{H\big(\|u_k\|\big)^{\frac{N}{N+2}}}.$$

From elliptic regularity (see (12)) we deduce

$$\|u_k\|_{W^{2,q}(\Omega)} \leq C\ \frac{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}}{\Big[H\big(\|u_k\|\big)\Big]^{\frac{N}{(N+2)q}}}.$$

Step 5. From Morrey's Theorem (see (13)), for any $x \in B(x_k, R_k)$

$$|u_k(x) - u_k(x_k)| \leq C\ (R_k)^{2-\frac{N}{q}}\ \frac{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}}{\Big[H\big(\|u_k\|\big)\Big]^{\frac{N}{(N+2)q}}}.$$

Particularizing $x = y_k$ in the above inequality and from (14) we obtain

$$C\ (R_k)^{2-\frac{N}{q}}\ \frac{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}}{\Big[H\big(\|u_k\|\big)\Big]^{\frac{N}{(N+2)q}}} \geq |u_k(y_k) - u_k(x_k)| = \frac{1}{2}\|u_k\|,$$

which implies

$$(R_k)^{2-\frac{N}{q}} \geq \frac{1}{2C}\ \frac{\|u_k\|\ \Big[H\big(\|u_k\|\big)\Big]^{\frac{N}{(N+2)q}}}{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}},$$

or equivalently

$$R_k \geq \left(\frac{1}{2C} \, \frac{\|u_k\| \, \Big[H\big(\|u_k\|\big)\Big]^{\frac{N}{(N+2)q}}}{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{1/\left(2-\frac{N}{q}\right)} . \tag{16}$$

Step 6. Consequently, taking into account (15), and that H is non-increasing,

$$\int_{B(x_k,R_k)} u_k f(u_k) H(u_k)\, dx \quad \geq \quad \frac{1}{2} \|u_k\| H(\|u_k\|) m_k \; \omega \, (R_k')^N,$$

where $\omega = \omega_N$ is the volume of the unit ball in $\mathbb{R}^N$.

Due to $B(x_k, R_k) \subset \Omega$, substituting inequality (16), and rearranging terms, we obtain

$$\begin{aligned}
&\int_\Omega u_k f(u_k) H(u_k)\, dx \\
&\geq \frac{1}{2} \|u_k\| H(\|u_k\|) m_k \; \omega \; \left(\frac{1}{2C} \, \frac{\|u_k\| \, \Big[H\big(\|u_k\|\big)\Big]^{\frac{N}{(N+2)q}}}{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{N}{2-\frac{N}{q}}} \\
&= C\, m_k \; \left(\Big[\|u_k\| H(\|u_k\|) \Big]^{\frac{2}{N}-\frac{1}{q}} \frac{\|u_k\| \, \Big[H\big(\|u_k\|\big)\Big]^{\frac{N}{(N+2)q}}}{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} \\
&= C\, m_k \; \left(\frac{\|u_k\|^{1+\frac{2}{N}-\frac{1}{q}} H\big(\|u_k\|\big)^{\frac{2}{N}-\frac{2}{(N+2)q}}}{{M_k}^{1-\frac{1}{q}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} \\
&= C\, \frac{m_k}{M_k} \; \left(\frac{\|u_k\|^{1+\frac{2}{N}-\frac{1}{q}} H\big(\|u_k\|\big)^{\frac{2}{N}-\frac{2}{(N+2)q}}}{{M_k}^{1-\frac{2}{N}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} .
\end{aligned}$$

At this moment, let us observe that from hypothesis (H1)' and (H2),

$$\frac{m_k}{M_k} \geq C, \qquad \text{for all} \quad k \qquad \text{big enough.}$$

Hence, taking again into account hypothesis (H2), and rearranging exponents, we can assert that

$$
\begin{aligned}
&\int_\Omega u_k f(u_k) H(u_k)\, dx \\
&\quad \geq C \left(\frac{\|u_k\|^{1+\frac{2}{N}-\frac{1}{q}} \Big[H\big(\|u_k\|\big)\Big]^{\frac{2}{N}-\frac{2}{(N+2)q}}}{M_k^{1-\frac{2}{N}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} \\
&\quad \geq C \left(\frac{\|u_k\|^{1+\frac{2}{N}-\frac{1}{q}} \Big[H\big(\|u_k\|\big)\Big]^{\frac{2}{N}-\frac{2}{(N+2)q}}}{\Big[f\big(\|u_k\|\big)\Big]^{1-\frac{2}{N}-\frac{1}{(2^\star-1)q}}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} \\
&\quad \geq C \left(\frac{\|u_k\|^{(N+2)\left[\frac{1}{N}-\frac{1}{(N+2)q}\right]} \Big[H\big(\|u_k\|\big)\Big]^{2\left[\frac{1}{N}-\frac{1}{(N+2)q}\right]}}{\Big[f\big(\|u_k\|\big)\Big]^{(N-2)\left[\frac{1}{N}-\frac{1}{(N+2)q}\right]}} \right)^{\frac{1}{\frac{2}{N}-\frac{1}{q}}} .
\end{aligned}
$$

Finally, we deduce

$$
\int_\Omega u_k f(u_k) H(u_k)\, dx \geq C \left(\frac{\|u_k\|^{2^*-1} \Big[H\big(\|u_k\|\big)\Big]^{\frac{2}{N-2}}}{f\big(\|u_k\|\big)} \right)^{\frac{(N-2)\left[\frac{1}{N}-\frac{1}{(N+2)q}\right]}{\frac{2}{N}-\frac{1}{q}}} ,
$$

and from hypothesis (H3.2),

$$
\left(\frac{\|u_k\|^{2^*-1} \Big[H\big(\|u_k\|\big)\Big]^{\frac{2}{N-2}}}{f\big(\|u_k\|\big)} \right) \to \infty \text{ as } k \to \infty,
$$

which contradicts (9), ending the proof. ☑

Next, we prove Theorem 2.1:

Proof of Theorem 2.1. Clearly hypotheses (H1) implies hypotheses (H1)'.

For non-convex domains, we use the Kelvin transform to get the *a-priori* bounds in a neighborhood of the boundary. Let us observe that we need additionally hypothesis (H1) (see Theorem A.12). All the other arguments work exactly in the same way as in the above proof. ☑

3. *A priori* estimates in a class of non-convex regions

In this Section we prove *a priori* bounds for the positive solutions to the boundary-value problem

$$\begin{cases} -\Delta u &= f(u), & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded C^2 domains with *convex-starlike* boundary, including *ring-like* regions, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a subcritical nonlinearity.

Let λ_1, ϕ_1 stand for the first eigenvalue, first eigenfunction, of the problem $-\Delta\phi_1 = \lambda_1\phi_1$ in Ω, $\phi_1 = 0$ on $\partial\Omega$.

Our main result is:

Theorem 3.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded C^2 domain with convex-starlike boundary. If the nonlinearity f is locally Lipschitzian and satisfies:*

(H1) *There exist contants $C_0 > 0$, $\beta_0 \in (0,1)$ such that* $\displaystyle\liminf_{s\to+\infty} \frac{\min_{[\beta_0 s, s]} f}{f(s)} \geq C_0.$

(H2) *There exists a constant $C_1 > 0$ such that* $\displaystyle\limsup_{s\to\infty} \frac{\max_{[0,s]} f}{f(s)} \leq C_1.$

(H3) *There exists a constant $C_2 > 0$ and a non-increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

(H3.1)

$$\liminf_{s\to+\infty} \frac{2NF(s) - (N-2)sf(s)}{sf(s)H(s)} \geq C_2 > 0,$$

and

(H3.2)

$$\lim_{s\to+\infty} \frac{f(s)}{s^{2^\star-1}\,\big[H(s)\big]^{\frac{2}{N-2}}} = 0.$$

(H4) $\displaystyle\liminf_{s\to+\infty} \frac{f(s)}{s} > \lambda_1$, *where λ_1 is the first eigenvalue of $-\Delta$ acting on $H_0^1(\Omega)$.*

Then there exists a uniform constant C, depending only on Ω and f, such that for every classical solution $u > 0$ to (17)*,*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Unlike results in [11] or [8], we do not assume $f(s)/s^{\frac{N+2}{N-2}}$ to be nonincreasing. The proof can be read in [8], we include it here by the sake of completeness.

Proof of Theorem 3.1. Step 1. Due to $n(x)\cdot x<0$ for all $x\in\Gamma_2$, we can choose $\varepsilon>0$ such that if $x\in\Gamma_1$ and $d(x,\Gamma_2)<\varepsilon$, then $n(x)\cdot x<0$. Let us define $\Gamma_1':=\Gamma_1\setminus\{x\in\partial\Omega : d(x,\Gamma_2)<\varepsilon\}$, and $\Gamma_2':=\partial\Omega\setminus\Gamma_1'$.

From now on, throughout this proof C denotes several constants independent of u. From 5 and de Giorgi-Nash type Theorems (see [20, Theorem 14.1]),

$$\|u\|_{C^{0,\alpha}(\omega_{7\delta/8}\setminus\omega_{\delta/8})}\le C,\qquad \text{for any}\quad \alpha\in(0,1),$$

where $\omega_t:=\{x\in\Omega\ :\ d(x,\Gamma_1')<t\}$.

From Schauder interior estimates (see [17, Theorem 6.2]),

$$\|u\|_{C^{1,\alpha}(\omega_{3\delta/4}\setminus\omega_{\delta/4})}\le C.$$

Finally, combining L^p estimates with Schauder boundary estimates (see [3], [17]),

$$\|u\|_{W^{2,p}(\omega_{\delta/2})}\le C,\qquad \text{for any}\quad p\in(1,\infty).$$

Consequently, there exists two constants $C,\delta>0$ independent of u, such that

$$\|u\|_{C^{1,\alpha}(\omega_{\delta})}\le C,\qquad \text{for any}\quad \alpha\in(0,1). \tag{18}$$

Step 2. Any classical solutions to (17) satisfies the following identity, known as *Pohozaev* identity (see [25]):

$$\int_\Omega NF(u)-\frac{N-2}{2}uf(u)=\int_{\partial\Omega}\left(x\cdot\nabla u\,\frac{\partial u}{\partial n}+\left[F(u)-\frac{1}{2}|\nabla u|^2\right]\,x\cdot n\right)d\sigma, \tag{19}$$

where $n(x)$ is the outward normal vector to the boundary at $x\in\partial\Omega$.

Since u vanishes on $\partial\Omega$, for any tangential vector $t(x)$ we have

$$t(x)\cdot\nabla u(x)=0,\qquad \text{for all } x\in\partial\Omega.$$

Moreover, since $\partial\Omega=\Gamma_1'\cap\Gamma_2'$ is a convex-starlike boundary, for each $x\in\Gamma_2'$, we have

$$x=s(x)n(x)+\tau(x),\qquad \text{where}\quad s(x)\le 0, \tag{20}$$

and $\tau(x)$ is tangential to $\partial\Omega$. In particular, (20) holds for any $x\in\Gamma_2'$.

Since $\frac{\partial u}{\partial n}(x):=\nabla u(x)\cdot n(x)$ and (20),

$$|\nabla u(x)|^2=\left(\frac{\partial u}{\partial n}\right)^2,\quad \text{and} \tag{21}$$

$$x\cdot\nabla u(x)=s(x)n(x)\cdot\nabla u(x)=s(x)\,\frac{\partial u}{\partial n}(x),\quad \text{for any } x\in\Gamma_2'.$$

Substituting $F(u(x)) = 0$ for all $x \in \partial\Omega$, and (20)-(21) in (19) we have

$$\begin{aligned}\int_\Omega \Big(NF(u) - \frac{N-2}{2} u f(u)\Big) dx &= \int_{\Gamma_1'} \Big[x \cdot \nabla u \,\frac{\partial u}{\partial n} - \frac{1}{2}|\nabla u|^2 \,(x \cdot n)\Big] d\sigma \\ &\quad + \int_{\Gamma_2'} \Big[x \cdot \nabla u \,\frac{\partial u}{\partial n} - \frac{1}{2}|\nabla u|^2 \,(x \cdot n)\Big] d\sigma \\ &= \int_{\Gamma_1'} \Big[x \cdot \nabla u \,\frac{\partial u}{\partial n} - \frac{1}{2}|\nabla u|^2 \,(x \cdot n)\Big] d\sigma \\ &\quad + \int_{\Gamma_2'} \frac{s(x)}{2} \Big(\frac{\partial u}{\partial n}\Big)^2 d\sigma.\end{aligned} \tag{22}$$

Also, since $s(x) \le 0$ for all $x \in \Gamma_2'$, from (22), and (18),

$$\int_\Omega \Big(NF(u) - \frac{N-2}{2} u f(u)\Big)\, dx \le C. \tag{23}$$

Next we prove that also

$$\int_\Omega \Big|NF(u) - \frac{N-2}{2} u f(u)\Big|\, dx \le C. \tag{24}$$

From hypothesis (H4), there exists a constant C such that if $s > C$, then $f(s) > 0$. From hypothesis (H3.1), in particular, there exists a constant C such that if $s > C$, then $2NF(s) - (N-2)sf(s) > 0$. Splitting the above integral in the set $S = \{x \in \Omega : |u| \le C\}$ and its complement $\Omega \setminus S$, since from (23) $\int_{\Omega\setminus S} \big(NF(u) - \frac{N-2}{2} u f(u)\big)\, dx \le C$, then (24) holds.

All other arguments work as in Theorems 2.1, 2.2 (see also [8]). ☑

A. Appendix I: The moving planes method, the Kelvin transform, and *a priori* bounds in a neighborhood of the boundary

In this Appendix, we collect some well known results on the moving planes method: Theorem A.1 and Theorem A.4. Next, we state results concerning a-priori bounds in a neighborhood of the boundary: Theorems A.8, and A.12. The remaining theorems indicates the arguments through the Kelvin transform, Theorem A.9 fix regions where a Kelvin transform of the solution has no critical points, and Theorems A.10, A.11 translate those results to the solution. All those results are essentially well known (see [11]); we include it here in order to clarify which hypotheses are used in the convex case and in the non-convex case.

A.1. The Kelvin transform

Let us recall that every C^2 domain Ω satisfies the following condition, known as the *uniform exterior sphere condition*:

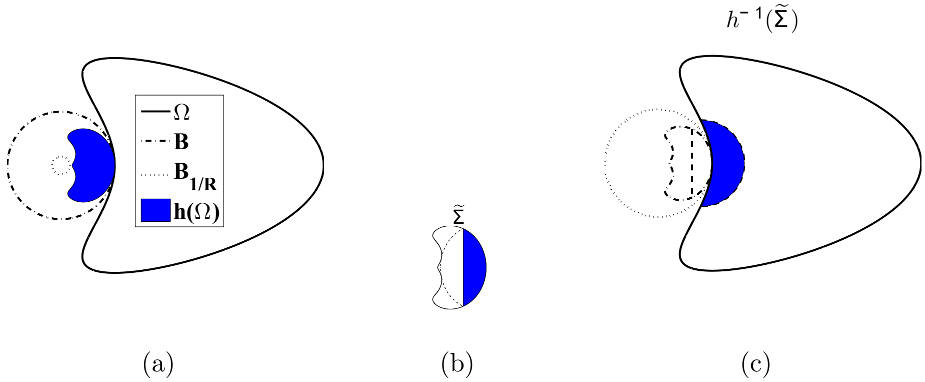


Figure 3. (a) The exterior tangent ball and the inversion of the boundary into the unit ball. (b) A maximal cap $\widetilde{\Sigma}$ in the transformed domain $h(\Omega)$. (c) The set $h^{-1}(\widetilde{\Sigma})$ (i.e., the inverse image of the maximal cap $\widetilde{\Sigma}$) in the original domain Ω.

(P) there exists a $\rho > 0$ such that for every $x \in \partial\Omega$ there exists a ball $B = B_\rho(y) \subset \mathbb{R}^N \setminus \Omega$ such that $\partial B \cap \partial\Omega = x$.

Let $x_0 \in \partial\Omega$, and let $\overline{B}$ be the closure of a ball intersecting $\overline{\Omega}$ only at the point x_0. Let us assume $x_0 = (1, 0, \cdots, 0)$, and B is the unit ball with center at the origin. The *inversion mapping*

$$x \to h(x) = \frac{x}{|x|^2} \tag{25}$$

is an homeomorphism from $\mathbb{R}^N \setminus \{0\}$ into itself; observe that $h(h(x)) = x$. We perform an inversion from Ω into the unit ball B, in terms of the inversion map $h|_\Omega$ (see fig. 3 (a)).

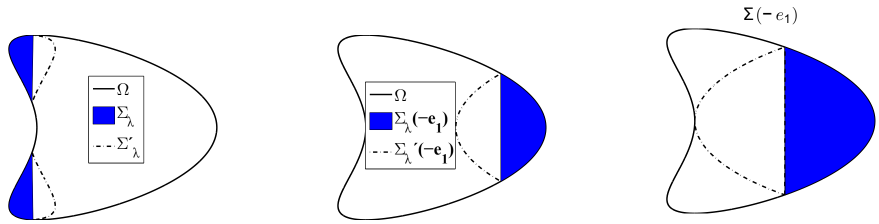
Let u solve (1). The *Kelvin transform* of u at the point $x_0 \in \partial\Omega$ is defined in the transformed domain $\widetilde{\Omega} := h(\Omega)$ by

$$v(y) := \left(\frac{1}{|y|}\right)^{N-2} u\left(\frac{y}{|y|^2}\right), \qquad \text{for} y \in \widetilde{\Omega}. \tag{26}$$

A.2. The moving planes method

We move planes in the x_1-direction to fix ideas. Let us first define some concepts and notations.

- The *moving plane* is defined in the following way: $T_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\}$;
- the *cap*: $\Sigma_\lambda := \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \cap \Omega \ : \ x_1 < \lambda\}$;
- the *reflected point*: $x^\lambda := (2\lambda - x_1, x')$;



(a) (b) (c)

Figure 4. (a) A cap Σ_λ and its reflected cap Σ'_λ in the e_1 direction. (b) A cap $\Sigma_\lambda(-e_1)$ and its reflected cap $\Sigma'_\lambda(-e_1)$ (in the $-e_1$ direction). (c) A maximal cap $\Sigma(-e_1)$.

- the *reflected cap*: $\Sigma'_\lambda := \{x^\lambda \ : \ x \in \Sigma_\lambda\}$ (see fig. 4(a));
- the minimum value for λ or starting value: $\lambda_0 := \min\{x_1 \ : \ x \in \overline{\Omega}\}$;
- the maximum value for λ: $\lambda^\star := \max\{\lambda \ : \ \Sigma'_\mu \subset \overline{\Omega} \quad \text{for all} \quad \mu \le \lambda\}$;
- the *maximal cap*: $\Sigma := \Sigma_{\lambda^\star}$.

The following Theorem is Theorem 2.1 in [14].

Theorem A.1. *Assume that f is locally Lipschitz, that Ω is bounded and that T_λ, x^λ, λ_0, $\lambda^\star$, Σ_λ Σ'_λ, and Σ are as above. If $u \in C^2(\overline{\Omega})$ satisfies* (1) *and $u > 0$ in Ω, then for any $\lambda \in (\lambda_0, \lambda^\star)$*

$$u(x) < u(x^\lambda) \qquad \textit{and} \qquad \frac{\partial u}{\partial x_1}(x) > 0 \qquad \textit{for all} \quad x \in \Sigma_\lambda.$$

Furthermore, if $\frac{\partial u}{\partial x_1}(x) = 0$ at some point in $\Omega \cap T_{\lambda^\star}$, then u is symmetric with respect to the the plane $T_{\lambda^\star}$, and $\Omega = \Sigma \cup \Sigma' \cup (T_{\lambda^\star} \cap \Omega)$.

Proof. See [14, Theorem 2.1 and Remark 1, p.219] for $f \in C^1$ and locally Lipschitzian respectively. ☑

Remark A.2. Set $x_0 \in \partial\Omega \cap T_{\lambda_0}$ (see fig. 4(a)). Let us observe that by definition of λ_0, T_{λ_0} is the tangent plane to the graph of the boundary at x_0, and the inward normal at x_0, is $n_i(x_0) = e_1$. The above Theorem says that the partial derivative following the direction given by the inward normal at the tangency point is strictly positive in the whole maximal cap. Consequently, *there are no critical points in the maximal cap.*

Now, we apply the above Theorem in any direction. According to the above Theorem, any positive solution of (1) satisfying (H1) has no stationary point in any maximal cap moving planes in any direction. This is the statement of the following Corollary. First, let us fix the notation for a general $\nu \in \mathbb{R}^N$ with $|\nu| = 1$. We set

- the *moving plane* defined as: $T_\lambda(\nu) = \{x \in \mathbb{R}^N \ : \ x \cdot \nu = \lambda\}$;

- the *cap*: $\Sigma_\lambda(\nu) = \{x \in \Omega \ : \ x \cdot \nu < \lambda\}$;
- the *reflected point*: $x^\lambda(\nu) = x + 2(\lambda - x \cdot \nu)\nu$;
- the *reflected cap*: $\Sigma'_\lambda(\nu) = \{x^\lambda \ : \ x \in \Sigma_\lambda(\nu)\}$, see fig. 4(b), for $\nu = -e_1$;
- the minimum value of λ: $\lambda_0(\nu) = \min\{x \cdot \nu \ : \ x \in \overline{\Omega}\}$;
- the maximum value of λ: $\lambda^\star(\nu) = \max\{\lambda \ : \ \Sigma'_\mu(\nu) \subset \overline{\Omega} \ \text{ for all } \mu \leq \lambda\}$;
- and the *maximal cap*: $\Sigma(\nu) = \Sigma_{\lambda^\star(\nu)}(\nu)$, see fig. 4(c), for $\nu = -e_1$.

Finally, let us also define the *optimal cap set*

$$\Omega^\bigstar = \bigcup_{\{\nu \in \mathbb{R}^N, |\nu|=1\}} \Sigma(\nu).$$

Applying Theorem A.1 in any direction, we can assert that there are not critical points in the union of all the maximal caps following any direction. The set $\Omega^\bigstar$ is the union of the maximal caps in any direction, and in particular, the maximum of a positive solution is attained in the complement of $\Omega^\bigstar$. Thus we have:

Corollary A.3. *Assume that f is locally Lipschitzian, that Ω is bounded, and that $\Omega^\bigstar$ is the optimal cap set defined as above.*
If $u \in C^2(\overline{\Omega})$ satisfies (1) *and $u > 0$ in Ω, then*

$$\max_{\overline{\Omega}} u = \max_{\overline{\Omega} \setminus \Omega^\bigstar} u.$$

If $\Omega^\bigstar$ is a boundary neighborhood of $\partial\Omega$ in $\overline{\Omega}$, as it happens in convex domains, then there is $\varepsilon > 0$ depending only on the geometry of Ω (independent of f and u) such that u has no stationary points in a ε-neighborhood of $\partial\Omega$. Next we study the case in which $\Omega^\bigstar$ is not a neighborhood of $\partial\Omega$ in Ω.

We prove that the maximum of u in the whole domain Ω can be bounded above by a constant multiplied by the maximum of u in some open set strongly contained in Ω (see Theorem A.11 below).

To achieve this result, we will need the moving plane method for a nonlinearity $f = f(x,u)$. Next we study this method on nonlinear equations in a more general setting. Let us consider the nonlinear equation

$$F\left(x, u, \nabla u, \left(\partial^2_{ij} u\right)_{i,j=1,\cdots,N}\right) = 0, \tag{27}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$ is a real function, $F = F(x,s,p,r)$ and $\partial^2_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$. The operator F is assumed to be elliptic, i.e., for positive constants $m, \ M$,

$$M|\xi|^2 \geq \sum_{i,j} \frac{\partial F}{\partial r_{ij}} \xi_i \xi_j \geq m|\xi|^2, \qquad \forall \xi \in \mathbb{R}^N.$$

On the function F we will assume:

(F1) F is continuous and differentiable with respect to the variables $s, p_i, r_{i,j}$, for all values of its arguments $(x, s, p, r) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$.

(F2) For all $x \in \partial\Omega \cap \{x_1 < \lambda^\star\}$, $F(x, 0, 0, 0)$ satisfies either

$$F(x, 0, 0, 0) \geq 0 \qquad \text{or} \quad F(x, 0, 0, 0) < 0.$$

(F3) F satisfies

$$F\left(x^\lambda, s, (-p_1, p'), \hat{r}\right) \geq F(x, s, p, r),$$

for all $\lambda \in [\lambda_0, \lambda^\star)$, $x \in \Sigma(\lambda)$ and $(s, p, r) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$ with $s > 0$ and $p_1 < 0$, where $p = (p_1, p') \in \mathbb{R} \times \mathbb{R}^{N-1}$, $\hat{r} = \begin{pmatrix} r_{11} & -r'_{1\cdot} \\ r_{21} & r'_{2\cdot} \\ \vdots & \vdots \\ r_{N1} & r'_{N\cdot} \end{pmatrix}$, and $r'_{i\cdot} := (r_{i2}, \cdots, r_{iN})$, for $i = 1, \cdots, N$.

The following theorem is Theorem 2.1' in [14].

Theorem A.4. *Assume that Ω is bounded and that T_λ, x^λ, λ_0, $\lambda^\star$, Σ_λ Σ'_λ, and Σ are as above. Let F satisfies conditions* (F1), (F2) *and* (F3).

If $u \in C^2(\overline{\Omega})$ satisfies (27) *and $u > 0$ in Ω, then for any $\lambda \in (\lambda_0, \lambda^\star)$*

$$u(x) < u(x^\lambda) \qquad \textit{and} \quad \frac{\partial u}{\partial x_1}(x) > 0 \qquad \textit{for all} \quad x \in \Sigma_\lambda.$$

Furthermore, if $\frac{\partial u}{\partial x_1}(x) = 0$ at some point in $\Omega \cap T_{\lambda^\star}$, then necessarily u is symmetric in the plane $T_{\lambda^\star}$, and $\Omega = \Sigma \cup \Sigma' \cup (T_{\lambda^\star} \cap \Omega)$.

As an immediate corollary in the semilinear situation we have the following one.

Corollary A.5. *Suppose $u \in C^2(\overline{\Omega})$ is a positive solution of*

$$-\Delta u = f(x, u), \quad \textit{in } \Omega, \qquad u = 0, \quad \textit{on } \partial\Omega. \tag{28}$$

Assume $f = f(x, s)$ and its first derivative f_s are continuous, for $(x, s) \in \overline{\Omega} \times \mathbb{R}$.

Assume that

$$f(x^\lambda, s) \geq f(x, s) \qquad \textit{for all} \quad x \in \Sigma(\lambda^\star), \qquad \textit{for all} \quad s > 0.$$

Then for any $\lambda \in (\lambda_0, \lambda^\star)$

$$u(x) < u(x^\lambda) \qquad \textit{and} \quad \frac{\partial u}{\partial x_1}(x) > 0 \qquad \textit{for all} \quad x \in \Sigma_\lambda.$$

Furthermore, if $\frac{\partial u}{\partial x_1}(x) = 0$ at some point in $\Omega \cap T_{\lambda^\star}$, then necessarily u is symmetric in the plane $T_{\lambda^\star}$, and $\Omega = \Sigma \cup \Sigma' \cup (T_{\lambda^\star} \cap \Omega)$.

Set $x_0 \in \partial\Omega \cap T_{\lambda_0}$. The above Theorem says that the partial derivative following the direction given by the inward normal, $n_i(x_0)$, at the tangency point x_0, is strictly positive in the whole maximal cap $\Sigma = \Sigma(n_i(x_0))$; consequently, the function $g(t) := u(x_0 + t\, n_i(x_0))$ is non-decreasing for $t \in [0, t_0]$ for some $t_0 = t_0(x_0) > 0$.

Now consider a neighborhood of x_0 denoted by $B_{\delta_0}(x_0)$. We can observe that for any $x \in \partial\Omega \cap B_{\delta_0}(x_0) \cap \Sigma$, also the function $g(t) := u(x + t\, n_i(x_0))$ is non-decreasing for $t \in [0, t_0]$ for some $t_0 = t_0(x_0, x) > 0$. By choosing points x such that $dist(x, T_{\lambda^*}(n_i(x_0))) > \delta$, we see that the function $g(t) := u(x + t\, n_i(x_0))$ is non-decreasing for $t \in [0, \delta]$ for any $x \in \partial\Omega \cap \Sigma(n_i(x_0)) : dist(x, T_{\lambda^*}(n_i(x_0))) > \delta$.

Now, let us move to a different cap, in a neighborhood of x_0. We apply the idea, to their corresponding maximal caps Σ, with their corresponding vectors ν. Then, choosing points in the intersection of the maximal caps, such that $dist(x, T_\lambda(\nu)) > \delta$, also the function $g(t) := u(x + t\nu)$ is increasing for $t \in [0, \delta]$. This is the statement of the following two corollaries, whose ideas are contained in [11].

Corollary A.6. *Assume that Ω is bounded and that $T_\lambda(\nu)$, $x^\lambda(\nu)$, $\lambda_0(\nu)$, $\lambda^\star(\nu)$, $\Sigma_\lambda(\nu)$ $\Sigma'_\lambda(\nu)$, and $\Sigma(\nu)$ are as above.*

Suppose $u \in C^2(\overline{\Omega})$ is a positive solution of (28). *Assume $f = f(x,s)$ and its first derivative f_s are continuous, for $(x,s) \in \overline{\Omega} \times \mathbb{R}$.*

Let $x_0 \in \partial\Omega$ such that $\Sigma = \Sigma(n_i(x_0)) \neq \emptyset$. Assume also that there exists a $\mu > 0$ such that

$$f(x^\lambda(\nu), s) \geq f(x,s) \quad \text{for all } x \in \Sigma(\nu) = \Sigma_{\lambda^\star(\nu)}(\nu), \text{ for all } s > 0,$$

where $\nu \in \mathbb{R}^N$ is such that $|\nu| = 1$, and $\nu \cdot n_i(x_0) \geq \mu$.

Then, there exists $\delta > 0$ depending only on the geometry of Ω, independent of f and u, such that the following holds:

the function

$$g(t) := u(x + t\nu) \quad \text{is non decreasing} \quad \text{for any } t \in [0, \delta],$$

for any $\nu \in \mathbb{R}^N$, such that $|\nu| = 1$, $\nu \cdot n_i(x_0) \geq \mu$, and for any $x \in \partial\Omega$ such that

$$x \in \bigcap_{\nu \cdot n_i(x_0) \geq \mu} \{\Sigma(\nu) : dist(x_0, T_{\lambda^*}(\nu)) > \delta\}.$$

For each point in a $\delta/2$ neighborhood of the boundary, there exists a cone K depending on the point, such that the function at that point is less or equal than the function at any point of the cone K. Now, we can choose a subset $K' \subset K$ depending on the point, but whose measure can be made independent of the point; remember that the function at that point is still less or equal than the function at any point of the subset K'. This is the statement of the following corollary, whose ideas, as we already said, are included in [11].

Corollary A.7. *Assume that Ω is bounded and that x_0, ν, $T_\lambda(\nu)$, $x^\lambda(\nu)$, $\lambda_0(\nu)$, $\lambda^\star(\nu)$, $\Sigma_\lambda(\nu)$ $\Sigma'_\lambda(\nu)$, and $\Sigma(\nu)$ are as above. Assume all the hypothesis of Corollary A.6 holds. Let $\delta > 0$ be as described in Corollary A.6.*

Then, for any $x_1 = x + t_1\,\nu$ *with* $0 < t_1 < \delta/2$, *the function*

$$g(t) := u(x_1 + t\nu) \quad \textit{is non decreasing} \quad \textit{for any } t \in [0, \delta/2],$$

for any $\nu \in \mathbb{R}^N$, *such that* $|\nu| = 1$, $\nu \cdot n_i(x_0) \geq \mu$, *and for any* $x \in \partial\Omega$ *such that*

$$x \in \bigcap_{\nu \cdot n_i(x_0) \geq \mu} \{\Sigma(\nu) : dist(x_0, T_{\lambda^*}(\nu)) > \delta\}.$$

Moreover, there exists a positive number γ *(depending only on the geometry of* Ω, *and independent of* f *and* u*), such that:*

for any $x_1 = x + t_1\,\nu$ *with* $0 < t_1 < \delta/2$, *there exists a cone with vertex* x_1, $K = K(x_1) \subset \{x \in \Omega : t_1 < dist(x_0, \partial\Omega) < \delta\}$, *and a piece of that cone* $K' = K'(x_1)$ *such that*

(i) $meas(K'(x_1)) \geq \gamma > 0$;

(ii) $K'(x_1) \subset \{x \in \Omega : \delta/2 < dist(x_0, \partial\Omega) < \delta\}$;

(iii) $u(x_1) \leq u(x), \qquad$ *for any* $\quad x \in K$.

A.3. A priori bounds in a neighborhood of the boundary

From now on, the arguments split into two ways, depending on the convexity of the domain. If Ω is convex, we observe that, reasoning as in [11], specifically, using Corollary A.6 and Corollary A.7, any positive solution u is locally increasing in the maximal cap following directions close to the normal direction, which provides L^∞ bounds locally in a neighborhood of the boundary. This is the statement of the following Theorem.

Theorem A.8. *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded, convex domain with* C^2 *boundary. Assume that the nonlinearity* f *satisfy* (H4).

If $u \in C^2(\overline{\Omega})$ *satisfies* (1) *and* $u > 0$ *in* Ω, *then there exists a constant* $\delta > 0$ *depending only on* Ω *and not on* f *or* u, *and a constant* C *depending only on* Ω *and* f *but not on* u, *such that*

$$\max_{\Omega \setminus \Omega_\delta} u \leq C, \tag{29}$$

where $\Omega_\delta := \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}$.

Proof. As observed in [4], [11, p. 44], [23], [27], under hypothesis (H4), there exists a constant $C_1 > 0$ such that

$$\int_\Omega u\phi_1 \leq C_1 \qquad \int_\Omega f(u)\phi_1 \leq C_1, \tag{30}$$

for any u solving (1).

Next, we will use Corollary 3.7. Let us fix an arbitrary $x_0 \in \partial\Omega$ and let $n_i(x_0)$ be the inward normal at the boundary point x_0. Choose any $\nu \in \mathbb{R}^N$ such that $|\nu| = 1$, and $\nu \cdot n_i(x_0) \geq \mu$, for some $\mu > 0$ fixed. From Corollary 3.6, there exists a $\delta > 0$ depending only on the geometry of Ω, and independent of f and u, such that the function

$g(t) := u(x + t\nu)$ is non decreasing for any $t \in [0, \delta]$, and for any $x \in \partial\Omega$ in a certain neighborhood of x_0. The neighborhood of x_0 depends only on the convexity of Ω, and it is independent of f and u.

Taking into account that all the hypotheses of the mentioned Corollary 3.7 hold, and using specifically Corollary 3.7 (iii), we deduce that for any $x_1 = x+t_1\,\nu \in \Omega$, with $0 < t_1 < \delta/2$, there exists a cone with vertex x_1, $K = K(x_1) \subset \{y \in \Omega : t_1 < d(y, \partial\Omega) < \delta\}$ and a piece ot that cone $K' = K'(x_1) \subset \{y \in K(x_1) : \delta/2 < d(y, \partial\Omega) < \delta\}$ such that $|K'| \geq \gamma > 0$, and

$$u(x_1) \leq u(x), \qquad \text{for any} \quad x \in K'. \tag{31}$$

Taking into account (30), (31), and Corollary 3.7 (i), we deduce that

$$C_1 \geq \int_\Omega u\phi_1 \geq \int_{K'} u\phi_1 \geq u(x_1) \int_{K'} \phi_1 \geq u(x_1)\,\gamma \min_{\Omega_{\delta/2}} \phi_1.$$

Consequently, there exists a constant C only dependent on f and on the geometry of Ω such that

$$u(x_1) \leq C \qquad \text{for all} \quad x_1 \in \Omega \setminus \Omega_{\delta/2}.$$

Then there exists a constant $\delta > 0$, depending only on Ω and not on f or u, and a constant C depending only on Ω and f but not on u, such that (29) holds. ☑

Next, we go through the non-convex case, reasoning on the Kelvin transform. First, in Theorem A.9, we fix regions where a Kelvin transform of the solution has no critical points. This is the statement of the following theorem, whose ideas are contained in [11]. Let us fix some notation. For any $x_0 \in \partial\Omega$, let $\tilde{n}_i(x_0)$ be the inward normal at x_0 in the transformed domain $\widetilde{\Omega} = h(\Omega)$, where h is defined in (25), and let $\widetilde{\Sigma} = \widetilde{\Sigma}(\tilde{n}_i(x_0))$ be its maximal cap (see fig. 3(b)).

Theorem A.9. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary. Assume that the nonlinearity f satisfies* (H1)*.*

If $u \in C^2(\overline{\Omega})$ satisfies (1) *and $u > 0$ in Ω, then for any $x_0 \in \partial\Omega$ its maximal cap in the transformed domain $\widetilde{\Sigma}$ is nonempty, and its Kelvin transform v, defined by* (26)*, has no critical point in the maximal cap $\widetilde{\Sigma}$.*

Consequently, for any $x_0 \in \partial\Omega$, there exists a $\delta > 0$ only dependent of Ω and x_0, and independent of f and u, such that its Kelvin transform v has no critical point in the set $B_\delta(x_0) \cap h(\Omega)$.

Proof. Since Ω is a C^2 domain, it satisfies a uniform exterior sphere condition (P). Let $x_0 \in \partial\Omega$, and let $\overline{B}$ be the closure of a ball intersecting $\overline{\Omega}$ only at the point x_0. For convenience, by scaling, translating and rotating the axes, we may assume that $x_0 = (1, 0, \cdots, 0)$, and B is the unit ball with center at the origin.

We perform an inversion h from Ω into the unit ball B, by using the inversion map $x \to h(x) = \frac{x}{|x|^2}$. Due to $\overline{B} \cap \overline{\Omega} = \{x_0\}$, and to the boundedness of Ω, there exists some $R > 0$ such that

$$1 \leq |x| \leq R \qquad \text{for any} \quad x \in \Omega, \tag{32}$$

and the image

$$\widetilde{\Omega} = h(\Omega) = \left\{ y = h(x) \in \mathbb{R}^N \ : \ x = \frac{y}{|y|^2} \in \Omega \right\} \subset B \setminus B_{1/R}.$$

Note that $0 \notin h(\Omega)$ (see fig. 3(a)). Moreover, $\widetilde{\Omega}$ is strictly convex near x_0 and the maximal cap $\widetilde{\Sigma} = \widetilde{\Sigma}(\tilde{n}_i(x_0))$ contains a full neighborhood of x_0 in $\widetilde{\Omega}$, where $\tilde{n}_i(x_0)$ is the normal inward at x_0 (see lemma B.1 in the Appendix; see also fig. 3(b)). Observe that, by construction $\tilde{n}_i(x_0) = -e_1$.

Next, we consider the Kelvin transform of the solution defined by (26). The function v is well defined on $h(\Omega)$, and writing $r = |x|$, $\omega = \frac{x}{|x|}$ and Δ_ω for the Laplace-Beltrami operator on ∂B_1, the function v satisfies

$$\begin{aligned}
\Delta v(r,\omega) &= \left[\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_\omega\right] v(r,\omega) \\
&= \left[\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_\omega\right] \left(\frac{1}{r}\right)^{N-2} u\big(\tfrac{1}{r},\omega\big) \\
&= \frac{1}{r^{N-1}}\frac{\partial}{\partial r} r^{N-1}\frac{\partial}{\partial r}\left[\left(\frac{1}{r}\right)^{N-2} u\big(\tfrac{1}{r},\omega\big)\right] + \frac{1}{r^N}\Delta_\omega u \\
&= \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left[-(N-2)u\big(\tfrac{1}{r},\omega\big) - \frac{1}{r}u_r\big(\tfrac{1}{r},\omega\big)\right] + \frac{1}{r^N}\Delta_\omega u \\
&= \frac{1}{r^{N-1}}\left[\frac{(N-2)}{r^2}u_r + \frac{1}{r^2}u_r + \frac{1}{r^3}u_{rr}\right] + \frac{1}{r^N}\Delta_\omega u \\
&= \frac{1}{r^{N+2}}\left[u_{rr} + \frac{N-1}{1/r}u_r + \frac{1}{1/r^2}\Delta_\omega u\right] = \frac{1}{r^{N+2}}\Delta u\big(\tfrac{1}{r},\omega\big).
\end{aligned}$$

Therefore, $v > 0$ in $\widetilde{\Omega}$ satisfies

$$-\Delta v(y) = \frac{1}{|y|^{N+2}} f\left(|y|^{N-2}v(y)\right), \quad \text{in } \widetilde{\Omega}, \qquad v = 0, \quad \text{on } \partial\widetilde{\Omega}.$$

From hypothesis (H1), we see that the function

$$g(y,s) = \frac{1}{|y|^{N+2}} f\left(|y|^{N-2}s\right) \tag{33}$$

satisfies the hypothesis of Corollary A.5. By construction, it is straightforward that $|y^\lambda| < |y|$ for all $y \in \widetilde{\Sigma}$ (see fig. 3 (a) and (b), and remain that the origin is at the center of the ball B). By (H1),

$$g(y^\lambda, s) \geq g(y,s) \qquad \text{for all} \quad y \in \widetilde{\Sigma},$$

where $\widetilde{\Sigma}$ is the maximal cap in the transformed domain (see fig. 3 (b)). Therefore, the hypotheses of Corollary A.5 are fulfilled, and hence v has no critical point in the maximal cap $\widetilde{\Sigma}$, which completes the proof choosing δ such that $B_\delta(x_0) \cap h(\Omega) \subset \widetilde{\Sigma}$. ☑

We are now ready to state the following theorem, essentially contained in [11]. This result is composed of two theorems: the first one, Theorem A.10 below, is the local version in a neighborhood of a boundary point; the second one, Theorem A.11, is the global version.

Theorem A.10. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary. Assume that the nonlinearity f satisfies* (H1)*.*

If $u \in C^2(\overline{\Omega})$ satisfies (1) *and $u > 0$ in Ω, then for any $x_0 \in \partial\Omega$ there exists a $\delta > 0$ only dependent of Ω and x_0, and independent of f and u such that*

$$\max_{\Omega} u \leq C \max_{\Omega \setminus B_\delta(x_0)} u.$$

The constant C depends on Ω but not on x_0, f or u.

Proof. Let $x_0 \in \partial\Omega$; if there exists a $\delta > 0$ such that $B_\delta(x_0) \cap \Omega \subset \Omega^\star$, (as it happens in convex sets), the proof follows from Theorem A.9. We concentrate our attention in the complementary set.

Let $x_0 \in \partial\Omega$, and let $\overline{B}$ be the closure of a ball intersecting $\overline{\Omega}$ only at the point x_0. Let v be as defined in (26) for $y \in \widetilde{\Omega} = h(\Omega)$. By a direct application of Theorem A.9, v has no critical point in the maximal cap $\widetilde{\Sigma}$, and therefore

$$\max_{\widetilde{\Omega}} v(y) = \max_{\widetilde{\Omega} \setminus \widetilde{\Sigma}} v(y).$$

From definition of v, see (26), we obtain that

$$\max_{\Omega} |x|^{N-2} u(x) = \max_{\Omega \setminus h^{-1}(\widetilde{\Sigma})} |x|^{N-2} u(x),$$

where $h^{-1}(\widetilde{\Sigma})$ is the inverse image of the maximal cap (see fig 3(b)-(c)). Due to the boundedness of Ω (see (32)), we deduce

$$\max_{\Omega} u(x) \leq R^{N-2} \max_{\Omega \setminus h^{-1}(\widetilde{\Sigma})} u(x),$$

which concludes the proof choosing $C = R^{N-2}$ and δ such that $B_\delta(x_0) \subset h^{-1}(\widetilde{\Sigma})$, and therefore $\Omega \setminus h^{-1}(\widetilde{\Sigma}) \subset \Omega \setminus B_\delta(x_0)$. ☑

The following Theorem is just a compactification process of the above result.

Theorem A.11. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary. Assume that the nonlinearity f satisfies* (H1)*.*

If $u \in C^2(\overline{\Omega})$ satisfies (1) *and $u > 0$ in Ω, then there exists two constants C and δ depending only on Ω and not on f or u such that*

$$\max_{\Omega} u \leq C \max_{\Omega_\delta} u,$$

where $\Omega_\delta := \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}$.

Proof. Since Ω is a C^2 domain, it satisfies a uniform exterior sphere condition (P). Thanks to that property, we can choose a constant $C = (R/\rho)^{N-2}$ satisfying the above inequality.

Moreover, let us note that from Theorems A.9 and A.10, the constant δ only depends on geometric properties of the domain Ω. ☑

Finally, we observe that, reasoning as in [11] on the Kelvin transform, specifically using Corollary A.6 and Corollary A.7, the Kelvin transform of u at $x_0 \in \partial\Omega$ is locally increasing in the maximal cap of the transformed domain, which provides L^∞ bounds for the Kelvin transform locally. By a compactification process, we then translate this into L^∞ bounds in a neighborhood of the boundary for any solution of the elliptic equation. This is the statement of the following theorem.

Theorem A.12. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary. Assume that the nonlinearity f satisfies* (H1) *and* (H4).

If $u \in C^2(\overline{\Omega})$ satisfies (1) *and $u > 0$ in Ω, then there exists a constant $\delta > 0$ depending only on Ω and not on f or u, and a constants C depending only on Ω and f but not on u, such that*

$$\max_{\Omega\setminus\Omega_\delta} u \leq C, \tag{34}$$

where $\Omega_\delta := \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}$.

Proof. We shall reason as in the proof of Theorem A.8. As observed in [4], [11, p. 44], [23], [27], under hypothesis (H4), there exists a constant $C_1 > 0$ such that

$$\int_\Omega u\phi_1 \leq C_1 \qquad \int_\Omega f(u)\phi_1 \leq C_1, \tag{35}$$

for any u solving (1).

Let us fix an arbitrary $x_0 \in \partial\Omega$ and consider the Kelvin transform of u at the point $x_0 \in \partial\Omega$, denoted by $v = v(x_0)$.

Next, we use Corollary A.7 on the Kelvin transform. We only need to note that, by construction, it is straightforward that there exists a $\tilde{\mu} > 0$ such that for any $\nu \in \mathbb{R}^N$ such that $|\nu| = 1$ and $\nu \cdot \tilde{n}_i(x_0) \geq \tilde{\mu}$, (observe that $\tilde{n}_i(x_0) = n_e(x_0)$), the following holds:

$$|y^\lambda(\nu)| \leq |y| \qquad \text{for all} \quad y \in \bigcap_{\nu\cdot\tilde{n}_i(x_0)\geq\tilde{\mu}} \widetilde{\Sigma}(\nu)$$

(see fig. 3 (a) and (b), and remember that the origin is at the center of the ball B); then, by (H1), and taking into account the definition of g (see (33)), we obtain

$$g(y^\lambda(\nu), s) \geq g(y, s) \qquad \text{for all } y \in \bigcap_{\nu\cdot\tilde{n}_i(x_0)\geq\tilde{\mu}} \widetilde{\Sigma}(\nu), \quad \text{for all } s > 0.$$

Therefore, all the hypothesis of Corollary A.7 hold. Now, using Corollary A.7 (iii), we deduce that there exist $\delta, \tilde{\delta} > 0$ only dependents on the geometry of Ω, such that for any $y_1 \in \widetilde{\Omega} \cap \widetilde{\Sigma}(\tilde{n}_i(x_0))$ with $d(y, \partial\widetilde{\Omega}) < \tilde{\delta}/2$, there exists a cone $\widetilde{K} = \widetilde{K}(y_1)$ and a subset $\widetilde{K}' \subset \widetilde{K}$ such that $|h^{-1}(\widetilde{K}')| \geq \gamma > 0$, $h^{-1}(\widetilde{K}') \subset \{x \in \Omega : \ \delta/2 < d(x, \partial\Omega) < \delta\}$, and

$$v(y_1) \leq v(y), \qquad \text{for any} \quad y \in \widetilde{K}'.$$

From definition of v, there exists a constant C only dependent on the geometry of Ω such that

$$u(x_1) \leq Cu(x), \qquad \text{for any} \quad x \in h^{-1}(\widetilde{K}'), \tag{36}$$

where $x_1 = h^{-1}(y_1)$, $x = h^{-1}(y)$.

Taking into account (35), (36), and Corollary A.7 (i), we deduce that

$$C_1 \geq \int_\Omega u\phi_1 \geq \int_{h^{-1}(\widetilde{K}')} u\phi_1 \geq \frac{1}{C} u(x_1) \int_{h^{-1}(\widetilde{K}')} \phi_1 \geq \frac{\gamma}{C} u(x_1) \min_{\Omega_{\delta/2}} \phi_1.$$

Consequently, there exists a constant C only dependent on f and on the geometry of Ω such that

$$u(x_1) \leq C \qquad \text{for all} \quad x_1 \in h^{-1}\Big((\widetilde{\Omega} \setminus \widetilde{\Omega}_{\tilde{\delta}/2}) \cap \widetilde{\Sigma}(\tilde{n}_i(x_0)) \Big).$$

Now we move $x_0 \in \partial\Omega$ and consider their corresponding Kelvin transforms. By a compactification process, there exists a constant $\delta > 0$ depending only on Ω and not on f or u, and a constants C depending only on Ω and f but not on u, such that (34) holds. ☑

B. *On the maximal cap in the transformed domain through the inversion map*

In this Appendix we show that for any boundary point of a C^2 domain, the maximal cap in the transformed domain is nonempty. This is a known result, but we include it here by the shake of completeness.

This result could see m surprising in presence of highly oscillatory boundaries. For example, assume that the boundary of Ω includes $\Gamma_2 = \big\{(x, f(x)) : f(x) := 1 + x^5 \sin\left(\frac{1}{x}\right),\ x \in [-0.01, 0.01]\big\}$ (to visualize the scale, see in fig. 5(b) $\big\{ \left(x, x^5 \sin\left(\frac{1}{x}\right)\right),\ x \in [-0.01, 0.01]\big\}$).

Let $h(\Gamma_2)$ be the image through the inversion map into the unit ball B, and let Γ_3 be the arc of the boundary ∂B given by $\Gamma_3 = \{(x, g(x)) : g(x) := \sqrt{1-x^2},\ x \in [-0.01, 0.01]\}$ (see fig. 5(c)). At this scale, the oscillations are not appreciable. We plot in 5(d) the derivative of the "vertical" distance between the boundary Γ_2 and the ball, concretely we plot $f'(x) - g'(x)$ for $x \in [-0.01, 0.01]$. We plot in 5(e) the second derivative of the "vertical" distance between the boundary and the ball, which is $f''(x) - g''(x)$ for $x \in [-5 \cdot 10^{-4}, 5 \cdot 10^{-4}]$. Let us observe that this second derivative is strictly positive, and that $f''(0) - g''(0) = 1$. Consequently, the first derivative is strictly increasing, and therefore the "vertical" distance $f(x) - g(x)$ does not oscillate.

Moreover, let us consider the image through the inversion map of the straight line $y = 1$, i.e. $h(x, 1) = h\left(\{(x, 1),\ x \in [-0.01, 0.01]\}\right)$. In fig. 5(f)-(g) we plot the second coordinate of the difference $h(\Gamma_2) - h(x, 1)$. The oscillation phenomena is present here. In fig. 5(h) we plot the second coordinate of the difference $h(\Gamma_2) - h(\partial B)$. This difference does not oscillate.

In fig. 5(a) we draw the inversion of the boundary into the unit ball at an inflexion point; more precisely we set $\Gamma_1 := \big\{(x, f(x)) : f(x) = \frac{x^3}{2} + 1,\ x \in [-\pi/4, \pi/4]\big\}$, which has an inflexion point at $x = 0$.

Let h denote the inversion map defined in (25), and let $\widetilde{\Omega} = h(\Omega)$ denote the image through the inversion map into the ball B. For any $x_0 \in \partial\Omega$, let $\tilde{n}_i(x_0)$ be the normal inward at x_0 in the transformed domain $\widetilde{\Omega}$, and let $\widetilde{\Sigma} = \widetilde{\Sigma}(\tilde{n}_i(x_0))$ be its maximal cap (see fig. 3(b)).

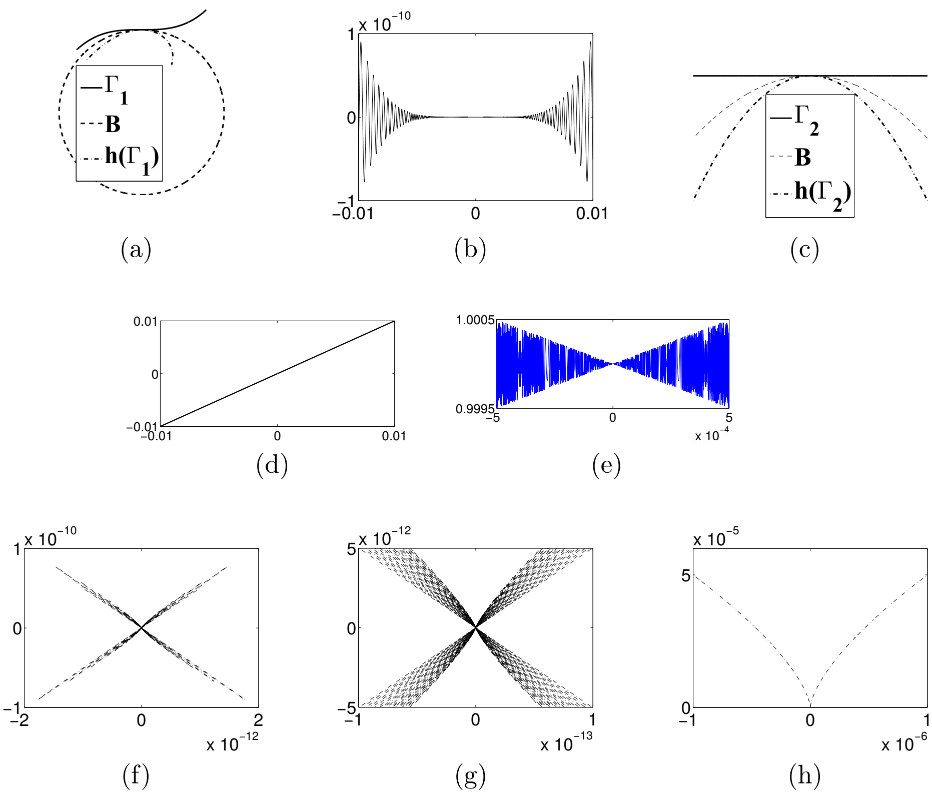


Figure 5. (a) An inflection point at the boundary Γ_1 joint with the inversion $h(\Gamma)$, and the unit circumference; (b) A degenerated critical point at the boundary Γ_2; (c) Γ_2 joint with its inversion into the unit ball, $h(\Gamma_2)$, and the arc of circumference, Γ_3; (d) $f'(x) - g'(x)$ for $x \in [-0.01, 0.01]$; (e) $f''(x) - g''(x)$ for $x \in [-5 \cdot 10^{-4}, 5 \cdot 10^{-4}]$; (f) Second coordinate of the difference $h(\Gamma_2) - h(x,1)$, where $h(x,1)$ is the image of the straight line $y = 1$; (g) a zoom of the same graphic; (h) Second coordinate of the difference $h(\Gamma_2) - h(\Gamma_3)$.

Lemma B.1. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary, then for any $x_0 \in \partial\Omega$, there exists a maximal cap $\widetilde{\Sigma} = \widetilde{\Sigma}(\tilde{n}_i(x_0))$ non empty.*

Proof. For convenience, we assume $x_0 = (0, \cdots, 0, 1)$, and B is the unit ball with center at the origin such that $\partial B \cap \partial\Omega = x_0$. Let $\{(x', \psi(x')); \|x\| < a\}$, $a > 0$, denote a parametrization of $\partial\Omega$ in a neighborhood of x_0. Hence,

$$\psi(0') = 1, \qquad \text{and} \quad \nabla_{N-1}\psi(0') = 0'. \tag{37}$$

Let $h(\Omega)$ stand for the image through the inversion map into the unit ball. From definition, $h(\partial\Omega \cap B(x_0))$ is given by

$$h\Big(x', \psi(x')\Big) = \frac{(x', \psi(x'))}{|x'|^2 + \psi(x')^2}, \qquad \text{for} \quad x' \in \mathcal{N}.$$

Set $y = h(x', \psi(x'))$ for $x' \in \mathcal{N}$ and with $y = (y', y_N)$. Since

$$y' = \frac{x'}{|x'|^2 + \psi(x')^2}, \; y_N = \frac{\psi(x')}{|x'|^2 + \psi(x')^2}, \text{ and } |y'|^2 + y_N^2 = \frac{1}{|x'|^2 + \psi(x')^2},$$

for $x' \in \mathcal{N}$, then $x' = \frac{y'}{|y'|^2 + y_N^2}$, for $y' \in \mathcal{N}'$, where $y' \in \mathcal{N}'$ if, and only if, $y' = \frac{x'}{|x'|^2 + \psi(x')^2}$ for some $x' \in \mathcal{N}$. Therefore,

$$y_N = \frac{\psi\left(\frac{y'}{|y'|^2 + y_N^2}\right)}{|y'|^2 + \psi\left(\frac{y'}{|y'|^2 + y_N^2}\right)^2}, \qquad \text{for} \quad y' \in \mathcal{N}',$$

and

$$h(\partial\Omega \cap B(x_0)) = \left\{(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \; F(y', y_N) = 0, \; y' \in \mathcal{N}'\right\},$$

where

$$F(y', y_N) := y_N \left[|y'|^2 + \psi\left(\frac{y'}{|y'|^2 + y_N^2}\right)^2\right] - \psi\left(\frac{y'}{|y'|^2 + y_N^2}\right). \tag{38}$$

Differentiating (38) with respect to y_N we obtain

$$\begin{aligned}\frac{\partial F}{\partial y_N}(y', y_N) = &\left[|y'|^2 + \psi\left(\frac{y'}{|y'|^2 + y_N^2}\right)^2\right] + y_N \frac{\partial}{\partial y_N}\left[|y'|^2 + \psi\left(\frac{y'}{|y'|^2 + y_N^2}\right)^2\right] \\ &- \sum_{i=1}^{N-1} \frac{\partial \psi}{\partial y_i}\left(\frac{y'}{|y'|^2 + y_N^2}\right) \frac{\partial}{\partial y_N}\left(\frac{y_i}{|y'|^2 + y_N^2}\right).\end{aligned}$$

Substituting at $(y', y_N) = (0', 1)$ and taking into account (37),

$$\begin{aligned}\frac{\partial F}{\partial y_N}(0', 1) = &\, 1 + 2\psi\left(\frac{y'}{|y'|^2 + y_N^2}\right) \sum_{i=1}^{N-1} \frac{\partial \psi}{\partial y_i}\left(\frac{y'}{|y'|^2 + y_N^2}\right) \frac{\partial}{\partial y_N}\left(\frac{y_i}{|y'|^2 + y_N^2}\right)\Bigg|_{(y', y_N) = (0', 1)} \\ =&1 \neq 0.\end{aligned}$$

Therefore, by the Implicit Function Theorem there exists an open neighborhood of $0'$, $B_\delta(0') \subset \mathbb{R}^{N-1}$, and a unique function $\phi : B_\delta(0') \to \mathbb{R}$, $\phi \in C^2(B_\delta(0'))$, such that $\phi(0') = 1$, and

$$F(y', \phi(y')) = 0 \qquad \text{for all} \quad y' \in B_\delta(0'). \tag{39}$$

Differentiating (39) with respect to y_j, $j = 1, \cdots, N-1$, using the chain rule and substituting at the point $(0', 1)$, we obtain

$$\frac{\partial F}{\partial y_j}(0', 1) + \frac{\partial F}{\partial y_N}(0', 1)\frac{\partial \phi}{\partial y_j}(0') = 0, \qquad \text{for} \quad j = 1, \cdots N-1. \tag{40}$$

On the other hand, differentiating (38) with respect to y_j and using the chain rule we obtain

$$\begin{aligned}\frac{\partial F}{\partial y_j}(y', y_N) =&y_N \frac{\partial}{\partial y_j}\left[|y'|^2 + \psi\left(\frac{y'}{|y'|^2 + y_N^2}\right)^2\right] \\ &- \sum_{i=1}^{N-1} \frac{\partial \psi}{\partial y_i}\left(\frac{y'}{|y'|^2 + y_N^2}\right) \frac{\partial}{\partial y_j}\left(\frac{y_i}{|y'|^2 + y_N^2}\right).\end{aligned}$$

Substituting at $(y', y_N) = (0', 1)$ and taking into account (37),

$$\frac{\partial F}{\partial y_j}(0',1) = 2\psi\left(\frac{y'}{|y'|^2+y_N^2}\right)\sum_{i=1}^{N-1}\frac{\partial\psi}{\partial y_i}\left(\frac{y'}{|y'|^2+y_N^2}\right)\frac{\partial}{\partial y_j}\left(\frac{y_i}{|y'|^2+y_N^2}\right)\Bigg|_{(y',y_N)=(0',1)}$$
$$=0.$$

Consequently, by (40)

$$\nabla_{N-1}\phi(0') = 0'. \tag{41}$$

Let us define

$$g(y') := \psi\left(\frac{y'}{|y'|^2+\phi(y')^2}\right), \text{ and } G(y') := \frac{g(y')}{|y'|^2+g(y')^2}, \text{ for } y' \in B_\delta(0').$$

By (37), $g(0') = 1$, and $G(0') = 1$. Moreover,

$$\left\{(y', y_N) \in \mathbb{R}^{N-1}\times\mathbb{R} :\ y_N = G(y'),\ y' \in B_\delta(0')\right\} \subset h(\partial\Omega)\cap B(x_0),$$

and

$$\left\{(y', y_N) \in \mathbb{R}^{N-1}\times\mathbb{R} :\ y_N < G(y'),\ y' \in B_\delta(0')\right\} \subset h(\Omega)\cap B(x_0).$$

Let us see that there exists $0 < \delta' \le \delta$ such that

$$U := \left\{(y', y_N) \in \mathbb{R}^{N-1}\times\mathbb{R} :\ y_N < G(y'),\ y' \in B_{\delta'}(0')\right\}$$

is a convex set. To achieve this, we use a characterization of convexity in the twice continuously differentiable case (see [13, p. 87-88]). *The set U is a convex set if, and only if, $D^2G(y')$ is negative semidefinite for all $y' \in B_\delta(0')$.* In fact, we will prove that $D^2G(0')$ is negative definite and by continuity, there exists some $\delta' > 0$ such that $D^2G(y')$ is negative semidefinite for all $y' \in B_{\delta'}(0')$. Differentiating,

$$\frac{\partial g}{\partial y_j} = \sum_{i=1}^{N-1}\frac{\partial\psi}{\partial y_i}\left(\frac{y'}{|y'|^2+\phi(y')^2}\right)\frac{\partial}{\partial y_j}\left(\frac{y_i}{|y'|^2+\phi(y')^2}\right),$$

and

$$\frac{\partial G}{\partial y_j} = \frac{\partial_j g}{|y'|^2+g(y')^2} - \frac{2g(y')\,(y_j+g\partial_j g)}{\left(|y'|^2+g(y')^2\right)^2}, \qquad \text{for} \quad j = 1,\cdots N-1,$$

where $\partial_j g = \frac{\partial g}{\partial y_j}$. Substituting at $y' = 0'$, and taking into account (37), we deduce

$$\nabla_{N-1}g(0') = 0', \qquad \text{and} \quad \nabla_{N-1}G(0') = 0'. \tag{42}$$

Taking second derivatives for $k = 1,\cdots N-1$, we obtain

$$\begin{aligned}\frac{\partial^2 g}{\partial y_k\partial y_j} \;=\; & \sum_{i=1}^{N-1}\frac{\partial}{\partial y_k}\left[\frac{\partial\psi}{\partial y_i}\left(\frac{y'}{|y'|^2+\phi(y')^2}\right)\right]\frac{\partial}{\partial y_j}\left(\frac{y_i}{|y'|^2+\phi(y')^2}\right)\\ & +\sum_{i=1}^{N-1}\frac{\partial\psi}{\partial y_i}\left(\frac{y'}{|y'|^2+\phi(y')^2}\right)\frac{\partial^2}{\partial y_k\partial y_j}\left(\frac{y_i}{|y'|^2+\phi(y')^2}\right),\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2 G}{\partial y_k \partial y_j} &= \frac{\partial^2_{kj} g}{|y'|^2 + g(y')^2} - \frac{2\partial_j g(y')\,(y_k + g\partial_k g)}{\left(|y'|^2 + g(y')^2\right)^2} \\
&\quad - \frac{2\partial_k g(y')\,(y_j + g\partial_j g) + 2g(y')\partial_k\,(y_j + g\partial_j g)}{\left(|y'|^2 + g(y')^2\right)^2} \\
&\quad + \frac{4g(y')\,(y_j + g\partial_j g)\,(y_k + g\partial_k g)}{\left(|y'|^2 + g(y')^2\right)^3},
\end{aligned}
$$

where $\partial^2_{kj} = \frac{\partial^2}{\partial y_k \partial y_j}$. Substituting at $y' = 0'$, and taking into account (37), we deduce

$$
\frac{\partial^2 g}{\partial y_k \partial y_j}(0') = \sum_{i=1}^{N-1} \frac{\partial}{\partial y_k}\left[\frac{\partial \psi}{\partial y_i}\left(\frac{y'}{|y'|^2 + \phi(y')^2}\right)\right] \frac{\partial}{\partial y_j}\left(\frac{y_i}{|y'|^2 + \phi(y')^2}\right)\Bigg|_{y'=0'}.
$$

Substituting at $y' = 0'$, and taking into account (42), we deduce

$$
\begin{aligned}
\frac{\partial^2 G}{\partial y_k \partial y_j}(0') &= \frac{\partial^2_{kj} g}{|y'|^2 + g(y')^2} - \frac{2g(y')\partial_k\,(y_j + g\partial_j g)}{\left(|y'|^2 + g(y')^2\right)^2}\Bigg|_{y'=0'} \\
&= \partial^2_{kj} g(0') - 2(\delta_{jk} + \partial^2_{kj} g(0')) = -2\delta_{jk} - \partial^2_{kj} g(0').
\end{aligned}
$$

Due to

$$
\frac{\partial}{\partial y_j}\left(\frac{y_i}{|y'|^2 + \phi(y')^2}\right) = \frac{\delta_{ij}}{|y'|^2 + \phi(y')^2} - \frac{2y_i\,(y_j + \phi\partial_j \phi)}{\left(|y'|^2 + g(y')^2\right)^2},
$$

where δ_{ij} is the Kronecker's delta, substituting at $y' = 0'$, and taking into account (41), we can write

$$
\frac{\partial}{\partial y_j}\left(\frac{y_i}{|y'|^2 + \phi(y')^2}\right)\Bigg|_{y'=0'} = \delta_{ij}. \tag{43}
$$

Moreover,

$$
\frac{\partial}{\partial y_k}\left[\frac{\partial \psi}{\partial y_i}\left(\frac{y'}{|y'|^2 + \phi(y')^2}\right)\right] = \sum_{m=1}^{N-1} \frac{\partial^2 \psi}{\partial y_m \partial y_i}\left(\frac{y'}{|y'|^2 + y_N^2}\right) \frac{\partial}{\partial y_k}\left(\frac{y_m}{|y'|^2 + y_N^2}\right);
$$

substituting at $y' = 0'$, and taking into account (43), we can write

$$
\frac{\partial}{\partial y_k}\left[\frac{\partial \psi}{\partial y_i}\left(\frac{y'}{|y'|^2 + \phi(y')^2}\right)\right]\Bigg|_{y'=0'} = \frac{\partial^2 \psi}{\partial y_k \partial y_i}(0').
$$

Let $A := \left(\partial^2_{kj}\psi(0')\right)_{j,k=1,\cdots N-1}$; then,

$$
\left(\partial^2_{kj} g(0')\right)_{j,k=1,\cdots N-1} = A, \quad \text{and} \quad \left(\partial^2_{kj} G(0')\right)_{j,k=1,\cdots N-1} = -(2I_{N-1} + A),
$$

where I_{N-1} is the identity matrix.

From hypothesis $\partial B \cap \partial\Omega = x_0$. Therefore the 'vertical' distance (distance in the x_N coordinate) between $\partial\Omega$ and ∂B is strictly positive, i.e.,

$$\psi(x') > \sqrt{1-|x'|^2} \text{ for all } x' \in \mathcal{N} \setminus 0' \quad \text{with } x = (x', x_N) \in \Omega \cap B(x_0),$$

or equivalently

$$[\psi(x')]^2 + |x'|^2 > 1 \text{ for all } x' \in \mathcal{N} \setminus 0' \quad \text{with } x = (x', x_N) \in \Omega \cap B(x_0).$$

Set $H(x') := [\psi(x')]^2 + |x'|^2$ for $x' \in \mathcal{N}$ with $x = (x', x_N) \in \Omega \cap B(x_0)$. Then $H(0') = 1$, and from the above inequality the point $x' = 0'$ is an strict minimum of the function H. Due to (37) every derivative of H evaluated at $0'$ is zero, and necessarily the Hessian matrix of H must be semi-positive definite, i.e.,

$$\left(\partial_k\psi\partial_j\psi + \psi\partial_{kj}^2\psi + \delta_{kj}\right)_{j,k=1,\cdots N-1}\Bigg|_{x'=0'} = A + I_{N-1},$$

is a semi-positive definite matrix. Hence the matrix $-(A + 2I_{N-1})$ is negative definite, and $y' = 0'$ is a strict maximum of the function G. As a consequence, there exists a $\delta' > 0$ such that the matrix $\left(\partial_{kj}^2 G(y')\right)_{j,k=1,\cdots N-1}$ is negative definite for all $y' \in B_{\delta'}(0')$. Consequently, the set U is a convex set.

Le us now choose $\gamma = \max\{G(y') \mid y' \in \partial B_{\delta'}(0')\}$. Due to $y' = 0'$ is a strict maximum of the function G, and that $G(0') = 1$, then $\gamma < 1$. The cap $\widetilde{\Sigma}_{(1-\gamma)/2}(-e_N)$ and its reflection $\widetilde{\Sigma}'_{(1-\gamma)/2}(-e_N)$ are non empty sets contained in $h(\Omega)$. Hence the maximal cap $\widetilde{\Sigma}$ contains $\widetilde{\Sigma}_{(1-\gamma)/2}(-e_N)$, which is nonempty, and concludes that the maximal cap $\widetilde{\Sigma}$ is a nonempty set. ☑

Acknowledgements

The author was partially supported by Grant MTM2016-75465, MINECO, Spain and Grupo de Investigación CADEDIF 920894, UCM.

References

[1] Atkinson F.V. and Peletier L.A., "Elliptic equations with nearly critical growth", *J. Differential Equations* 70 (1987), No.3, 349–365.

[2] Bahri A. and Coron J.M., "On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain", *Comm. Pure Appl. Math.* 41 (1988), No.3, 253–294.

[3] Brezis H., *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[4] Brezis H. and Turner R.E.L., "On a class of superlinear elliptic problems", *Comm. Partial Differential Equations* 2 (1977), No. 6, 601–614.

[5] Castro A. and Pardo R., "A priori bounds for positive solutions of subcritical elliptic equations", *Rev. Mat. Complut.* 28 (2015), No.3, 715–731.

[6] Castro A. and Pardo R., "Branches of positive solutions of subcritical elliptic equations in convex domains", *Dynamical Systems, Differential Equations and Applications, AIMS Proceedings* (2015), 230–238.

[7] Castro A. and Pardo R., "Branches of positive solutions for subcritical elliptic equations", *Progr. Nonlinear Differential Equations Appl: Contributions to Nonlinear Elliptic Equations and Systems* 86 (2015), 87–98.

[8] Castro A. and Pardo R., "A priori estimates for positive solutions to subcritical elliptic problems in a class of non-convex regions", *Discrete Contin. Dyn. Syst. Ser. B* 22 (2017), No.3, 783–790.

[9] Castro A. and Shivaji R., "Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric", *Comm. Partial Differential Equations* 14 (1989), No. 8-9, 1091–1100.

[10] Damascelli L. and Pardo R., "A priori estimates for some elliptic equations involving the p-Laplacian", N*onlinear Anal.* 41 (2018), 475–496.

[11] De Figueiredo D.G., Lions P.-L. and Nussbaum R.D., "A priori estimates and existence of positive solutions of semilinear elliptic equations", *J. Math. Pures Appl. (9)* 61 (1982), No. 1, 41–63.

[12] Ding W.Y., "Positive solutions of $\Delta u + u^{(n+2)/(n-2)} = 0$ on contractible domains", *J. Partial Differential Equations* 2 (1989), No.4, 83–88.

[13] Fenchel W., *Convex Cones, Sets and Functions*, Lecture Notes at Princeton University, Dept. of Mathematics, Princeton, N.J., 1953.

[14] Gidas B., Ni W.-M. and Nirenberg L., "Symmetry and related properties via the maximum principle", *Comm. Math. Phys.* 68 (1979), No. 3, 209-243.

[15] Gidas B., Ni W.-M. and Nirenberg L., "Symmetry of positive solutions of nonlinear elliptic equations in R^n", in Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud., 7, Academic Press (1981), 369–402.

[16] Gidas B. and Spruck J., "A priori bounds for positive solutions of nonlinear elliptic equations", *Comm. Partial Differential Equations* 6 (1981), No. 8, 883–901.

[17] Gilbarg D. and Trudinger N.S., *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[18] Han Z.-C., "Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent", *Ann. Inst. H. Poincaré Anal. Non Linéaire* 8 (1991), No.2, 159–174.

[19] Joseph D.D. and Lundgren T.S., "Quasilinear Dirichlet problems driven by positive sources", *Arch. Rational Mech. Anal.* 49 (1972/73), 241–269.

[20] Ladyzhenskaya O.A. and Ural'tseva N.N., *Linear and quasilinear elliptic equations*, Academic Press, New York-London, 1968.

[21] Lions P.-L., "On the existence of positive solutions of semilinear elliptic equations", *SIAM Rev.* 24 (1982), No. 4, 441–467

[22] Mavinga N. and Pardo R., "A priori bounds and existence of positive solutions for subcritical semilinear elliptic systems", *J. Math. Anal. Appl.* 449 (2017), No. 2, 1172–1188.

[23] Nussbaum R., "Positive solutions of nonlinear elliptic boundary value problems", *J. Math. Anal. Appl.* 51 (1975), No.2, 461–482.

[24] Pardo R. and Sanjuán A., "Asymptotics for positive radial solutions of elliptic equations approaching critical growth", Preprint.

[25] Pohozaev S.I., "On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$", *Dokl. Akad. Nauk SSSR* 165 (1965), 36–39.

[26] Serrin J., "A symmetry problem in potential theory", *Arch. Rational Mech. Anal.* 43 (1971), 304–318.

[27] Turner R.E.L., "A priori bounds for positive solutions of nonlinear elliptic equations in two variables", *Duke Math. J.* 41 (1974), 759-774.